Change point estimation for a stochastic heat equation

Statistics Seminar – University of Cambridge

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based on joint works with Markus Reiß, Claudia Strauch and Anton Tiepner

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University of Birmingham   Humboldt University of Berlin   Heidelberg University   Aarhus University

Some generalities on statistics for SPDEs

- Let A_ϑ be a self-adjoint generator of a C_0-semigroup on $L^2(\Lambda)$ for some domain $\Lambda \subset \mathbb{R}^d$ and consider the SPDE

$$\begin{cases} \mathrm{d}X(t) = A_\vartheta X(t)\,\mathrm{d}t + \mathrm{d}W(t), & t \in (0,T], \\ X(0) = X_0 \in L^2(\Lambda), & \\ X(t)|_{\partial\Lambda} = 0, & t \in (0,T], \end{cases}$$

  where W is a cylindrical Wiener process, that is $W(t) = \sum_{j \in \mathbb{N}} \beta_j(t) e_j$ for independent Brownian motions $(\beta_j)_j$ and a complete orthonormal system $(e_j)_j$ in $L^2(\Lambda)$

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which for $d > 1$ is typically only realised in a space of distributions
- the mild solution solves the SPDE in the sense
$$\langle X(t), z\rangle = \langle X_0, z\rangle + \int_0^t \langle X(s), A_\vartheta z\rangle\,\mathrm{d}s + \langle W(t), z\rangle, \quad z \in D(A_\vartheta).$$

- spectral observations: provided A_ϑ has an orthonormal eigenbasis (e_j) that is independent of ϑ (think $A_\vartheta = \vartheta\Delta$), observe ($j = 1, \dots, n$, $t \in [0, T]$)

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- observe $(\langle X(t), K_{\delta,i}\rangle)_{t\in[0,T], i=1,\dots,n}$, where for $K_{\delta,i} \in D(A_\vartheta)$,

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$\rightsquigarrow$ observations are generalised Itô processes (but not independent for $i \neq j$); asymptotics: $\delta \to 0$, n may be fixed or increase with δ^{-1}

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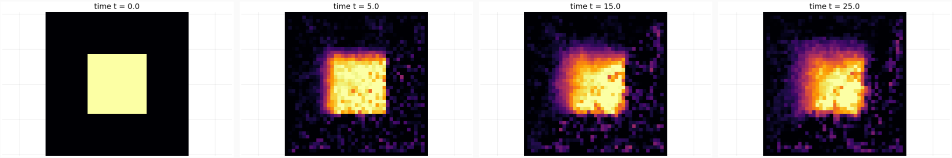
Change point model for stochastic heat equations



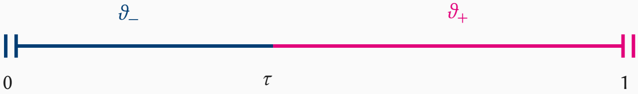
- Stochastic heat equation

$$\mathrm{d}X(t) = \Delta_\vartheta X(t)\,\mathrm{d}t + \mathrm{d}W(t), \qquad \Delta_\vartheta = \nabla \cdot \vartheta \nabla,$$

with Dirichlet boundary conditions, and broken diffusivity

$$\vartheta(x) = \vartheta_- \mathbf{1}_{\Lambda_-}(x) + \vartheta_+ \mathbf{1}_{\Lambda_+}(x), \qquad x \in [0,1]^d = \Lambda_- \uplus \Lambda_+, \vartheta_- \wedge \vartheta_+ > 0.$$

- special case for $d = 1$: $\Lambda_+ = (\tau, 1]$ with change point τ



The univariate case

$-\Delta_\vartheta$ is induced by Dirichlet form

$$\mathcal{E}(u, v) := \langle \vartheta \partial_x u, \partial_x v \rangle = \int_0^1 \vartheta(x)\, \partial_x u(x)\, \partial_x v(x)\, \mathrm{d}x, \quad u, v \in H_0^1((0,1)),$$

and generates C_0-semigroup $S_\vartheta(t) = \exp(t\Delta_\vartheta)$, $t \in [0, T]$, having transition densities that satisfy the heat kernel bound

$$p_t^\vartheta(x, y) \leq c_1 t^{-1/2} \exp\Big(- \frac{|x-y|^2}{c_2 t} \Big), \quad (x, y) \in (0,1)^2, t \in (0, T].$$

Mild solution

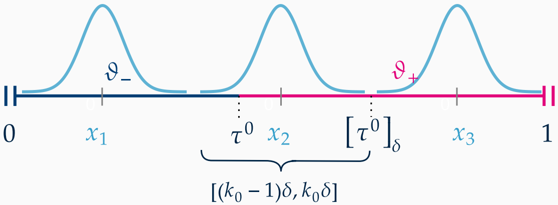
$$X(t) = \int_0^t S_\vartheta(t-s)\, \mathrm{d}W(s), \quad t \in [0, T], \qquad (\text{assume } X(0) \equiv 0)$$

is $L^2((0,1))$-valued and we have

$$\langle X(t), z \rangle = \int_0^t \langle X(s), \Delta_\vartheta z \rangle\, \mathrm{d}s + \langle W(t), z \rangle, \quad \forall\, z \in D(\Delta_\vartheta) = \big\{ u \in H_0^1((0,1)) : \vartheta \partial_x u \in H^1((0,1)) \big\}.$$

Observation model

- let $K: \mathbb{R} \to \mathbb{R}$ be a smooth kernel with supp $K \subset [-1/2, 1/2]$, $\|K\|_{L^2} = 1$ and for $\delta = n^{-1}$, $x_i = (i - 1/2)\delta\,(i \in \{1, \dots, \delta^{-1}\})$, define $K_{\delta,i}(x) = \delta^{-1/2} K(\delta^{-1}(x - x_i))$
- local observations $(X_{\delta,i}(t))_{t\in[0,T]} = (\langle X(t), K_{\delta,i}\rangle)_{t\in[0,T]}$ and $(X^{\Delta}_{\delta,i}(t))_{t\in[0,T]} = (\langle X(t), \Delta K_{\delta,i}\rangle)_{t\in[0,T]}$



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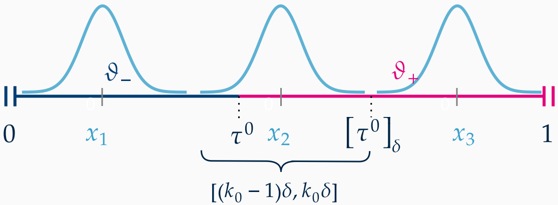

- we have

$$X_{\delta,i}(t)=\begin{cases}\int_0^t\vartheta^0_{\pm}X^{\Delta}_{\delta,i}(s)\,\mathrm{d}s+B_{\delta,i}(t), & i\gtrless k_0,\\ \int_0^t\int_0^s\langle\Delta_{\vartheta^0}S_{\vartheta^0}(s-u)K_{\delta,i},\mathrm{d}W(u)\rangle\,\mathrm{d}s+B_{\delta,k_0}(t), & i=k_0,\end{cases}$$

for independent Brownian motions $(B_{\delta,i})_{i\in[\delta^{-1}]}$

Estimation approach

- modified local log-likelihood:

$$\ell_{\delta,i}(\vartheta_-,\vartheta_+,\vartheta_\circ,k) := \vartheta_{\delta,i}(k)\int_0^T X_{\delta,i}^{\Delta}(t)\,\mathrm{d}X_{\delta,i}(t) - \frac{\vartheta_{\delta,i}(k)^2}{2}\int_0^T X_{\delta,i}^{\Delta}(t)^2\,\mathrm{d}t, \quad \vartheta_{\delta,i}(k) := \begin{cases}\vartheta_-, & i<k,\\ \vartheta_\circ, & i=k,\\ \vartheta_+, & i>k\end{cases}$$

- set $(\hat\vartheta_-,\hat\vartheta_+,\hat\vartheta_\circ,\hat\tau) := (\hat\vartheta_-,\hat\vartheta_+,\hat\vartheta_\circ,\hat k\delta)$, where

$$(\hat\vartheta_-,\hat\vartheta_+,\hat\vartheta_\circ,\hat k) := \underset{(\vartheta_-,\vartheta_+,\vartheta_\circ,k)}{\arg\max} \sum_{i\in[\delta^{-1}]} \ell_{\delta,i}(\vartheta_-,\vartheta_+,\vartheta_\circ,k)$$

$$= \underset{(\vartheta_-,\vartheta_+,\vartheta_\circ,k)}{\arg\min}\left\{\frac{1}{2}\sum_{i=1}^{\delta^{-1}}(\vartheta_{\delta,i}(k)-\vartheta_{\delta,i}^0)^2 I_{\delta,i} - \sum_{i=1}^{\delta^{-1}}(\vartheta_{\delta,i}(k)-\vartheta_{\delta,i}^0)M_{\delta,i} - \vartheta_{\delta,k_0}(k)R_{\delta,k_0}(\vartheta_\circ^0)\right\},$$

for

$$M_{\delta,i} := \int_0^T X_{\delta,i}^{\Delta}(t)\,\mathrm{d}B_{\delta,i}(t), \quad I_{\delta,i} := \int_0^T X_{\delta,i}^{\Delta}(t)^2\,\mathrm{d}t,$$

and $R_{\delta,k_0}(\vartheta_\circ^0)$ is an error term resulting from $K_{\delta,k_0} \notin D(\Delta_\vartheta)$ in general

Basic estimates

Lemma (Reiß, Strauch and T., 2023+)

- For any $i \in [\delta^{-1}] \setminus \{k_0\}$,
$$\mathbb{E}[I_{\delta,i}] = \frac{T}{2\vartheta(x_i)} \|K'\|_{L^2}^2 \delta^{-2} + \mathcal{O}(1),$$
  and, moreover, $\mathbb{E}[I_{\delta,k_0}] \sim \delta^{-2}$;
- for any vector $\alpha \in \mathbb{R}^n$ s.t. $\alpha_{k_0} = 0$,
$$\mathrm{Var}\Big(\sum_{i=1}^{\delta^{-1}} \alpha_i I_{\delta,i}\Big) \leq \frac{T}{2\underline{\vartheta}^3} \delta^{-2} \|\alpha\|_{\ell^2}^2 \|K'\|_{L^2}^2;$$
- $$\mathbb{E}[|R_{\delta,k_0}(\vartheta_\circ)|] \lesssim \delta^{-2}, \quad \mathrm{Var}(R_{\delta,k_0}(\vartheta_\circ)) \lesssim \delta^{-2},$$
  and, moreover,
$$\exists\, \vartheta_\circ^0 : \quad |\mathbb{E}[R_{\delta,k_0}(\vartheta_\circ^0)]| \leq \delta^{-1}.$$

Concentration results

$\sum_{i=1}^{\delta^{-1}} \alpha_i (I_{\delta,i} - \mathbb{E}[I_{\delta,i}])$ belongs to second Wiener chaos for an isonormal Gaussian process associated to $(X_i^\Delta(t))_{t\in[0,T], i\in[\delta^{-1}]}$ $\rightsquigarrow$ relate to Bernstein-type concentration result from Nourdin and Viens (2009)[2]

Proposition (Reiß, Strauch and T., 2023+)

Let $\alpha \in \mathbb{R}_+^n \setminus \{0\}$ s.t. $\alpha_{k_0} = 0$. Then, for any $z > 0$, we have

$$\mathbb{P}\Big(\Big|\sum_{i=1}^{n} \alpha_i (I_{\delta,i} - \mathbb{E}[I_{\delta,i}])\Big| \geq z\Big) \leq 2\exp\Big(-\frac{\underline{\vartheta}^2}{4\|\alpha\|_\infty} \frac{z^2}{z + \sum_{i=1}^n \alpha_i \mathbb{E}[I_{\delta,i}]}\Big).$$

[2]Nourdin, I., and F.G. Viens (2009). Density formula and concentration inequalities with Malliavin calculus. *Electron. J. Prob.*

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Proposition (Reiß, Strauch and T., 2023+)

Let

$$\overline{M}_{\delta,i} := \int_0^{\sigma_i} X_{\delta,i}^{\Delta}(t)\,\mathrm{d}B_{\delta,i}(t), \quad \text{where } \sigma_i := \inf\{t \geq 0 : \int_0^t X_{\delta,i}(s)^2\,\mathrm{d}s > \mathbb{E}[I_{\delta,i}].$$

Then, $(\overline{M}_{\delta,i})_{i \in [\delta^{-1}]} \sim N\big(0, \mathrm{diag}((\mathbb{E}[I_{\delta,i}])_{i \in [\delta^{-1}]})\big)$ and

$$\mathbb{P}\Big(\Big| \sum_{i=1}^n M_{\delta,i} - \overline{M}_{\delta,i} \Big| \geq z, \sum_{i=1}^n \alpha_i^2 |I_{\delta,i} - \mathbb{E}[I_{\delta,i}]| \leq L \Big) \leq \exp(-z^2/2L), \quad \alpha \in \mathbb{R}^n, z, L > 0$$

Rate of convergence

Define the jump height $\eta := \vartheta_+^0 - \vartheta_-^0$.

Theorem (Reiß, Strauch and T., 2023+)

Suppose that $\vartheta_\pm^0 \underset{\delta \to 0}{\longrightarrow} \vartheta_\pm^*$ and that $|\eta| \geq \underline{\eta} > 0$ for all $\delta \in 1/\mathbb{N}$. Then,

$$|\hat{\tau} - \tau^0| = \mathcal{O}_{\mathbb{P}}(\delta) \quad \text{and} \quad |\hat{\vartheta}_\pm - \vartheta_\pm^0| = \mathcal{O}_{\mathbb{P}}(\delta^{3/2}).$$

- the estimation rate for τ^0 cannot be improved due to discretisation effects
- the estimation rate for $\vartheta_\pm^0$ is the minimax optimal rate for parametric estimation from multiple local measurements in the model $A_\vartheta = \vartheta\Delta$ without change point[2]

[2] Altmeyer, R., Tiepner, A. and M. Wahl (2024). Optimal parameter estimation for linear SPDEs from multiple measurements. *Ann. Stat.*

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Proof outline:

1. verify basic consistency of $(\hat{\vartheta}_\pm, \hat{\tau})$
2. determine appropriate empirical process $(\mathcal{L}_\delta)_{\delta\in 1/\mathbb{N}}$ with $[\underline{\vartheta}, \overline{\vartheta}]^3 \times (0,1] \ni \chi \mapsto \mathcal{L}_\delta(\chi)$ such that
$$(\hat{\vartheta}_-, \hat{\vartheta}_+, \hat{\vartheta}_\circ, \hat{\tau}) \in \underset{\chi\in[\underline{\vartheta},\overline{\vartheta}]^3\times(0,1]}{\arg\min} \mathcal{L}_\delta(\chi)$$
3. control local fluctuations of centered empirical process $\mathcal{L}_\delta - \widetilde{\mathcal{L}}_\delta(\chi)$ around χ^0, where $\widetilde{\mathcal{L}}_\delta(\chi) = \mathbb{E}[\mathcal{L}_\delta(\chi)] + \mathcal{O}(\delta^2)$
4. exploit (non-standard) peeling device to prove convergence rate

Vanishing signal

- for the previous consistency result it was crucial that the jump height η does not vanish
- assume now that $\eta \underset{\delta\to 0}{\longrightarrow} 0$ and that $\vartheta^0_\pm = \vartheta^0_\pm(\delta)$ are known
- set $\hat\tau = \hat k \delta$, where

$$\begin{aligned}\hat k &:= \operatorname*{arg\,max}_{k=1,\dots,\delta^{-1}} \sum_{i=1}^{k}\left(\vartheta^0_- \int_0^T X^\Delta_{\delta,i}(t)\,\mathrm{d}X_{\delta,i}(t) - \frac{(\vartheta^0_-)^2}{2}\int_0^T X^\Delta_{\delta,i}(t)^2\,\mathrm{d}t\right)\\ &\qquad + \sum_{i=k+1}^{\delta^{-1}}\left(\vartheta^0_+ \int_0^T X^\Delta_{\delta,i}(t)\,\mathrm{d}X_{\delta,i}(t) - \frac{(\vartheta^0_+)^2}{2}\int_0^T X^\Delta_{\delta,i}(t)^2\,\mathrm{d}t\right)\\ &= \operatorname*{arg\,min}_{k=1,\dots,\delta^{-1}} Z_k,\end{aligned}$$

for

$$Z_k = \begin{cases} 0, & k = k_0,\\ -\eta\sum_{i=k+1}^{k_0}\int_0^T X^\Delta_{\delta,i}(t)\,\mathrm{d}B_{\delta,i}(t) + \frac{\eta^2}{2}\sum_{i=k+1}^{k_0}\int_0^T X^\Delta_{\delta,i}(t)^2\,\mathrm{d}t + \eta R_{\delta,k_0}(\vartheta^0_-), & k < k_0,\\ \eta\sum_{i=k_0+1}^{k}\int_0^T X^\Delta_{\delta,i}(t)\,\mathrm{d}B_{\delta,i}(t) + \frac{\eta^2}{2}\sum_{i=k_0+1}^{k}\int_0^T X^\Delta_{\delta,i}(t)^2\,\mathrm{d}t, & k > k_0,\end{cases}.$$

Limit theorem for vanishing signal

Reformulate the estimator again in terms of an M-estimator: Let $v_\delta \to 0$, and define

$$M_{T,\delta}^{\tau^0}(h) = M_{T,\delta}(\tau^0 + hv_\delta) - M_{T,\delta}(\tau^0), \quad \text{for } M_{T,\delta}(z) := \sum_{i=1}^{\lceil z/\delta \rceil} M_{\delta,i},\ z \in [0,1]$$

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s.t.

$$\mathcal{Z}_\delta(v_\delta^{-1}(\hat{\tau} - \tau^0)) = \min_{h \in [-\tau_0/v_\delta, (1-\tau^0)/v_\delta]} \mathcal{Z}_\delta(h) + \mathcal{O}_{\mathbb{P}}(\eta^2\delta^{-2}), \quad \text{for } \mathcal{Z}_\delta(h) := \eta M_{T,\delta}^{\tau^0}(h) + \frac{\eta^2}{2} I_{T,\delta}^{\tau^0}(h)$$

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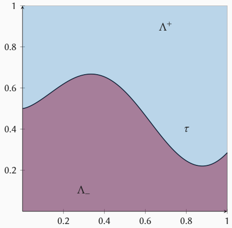
Theorem (Reiß, Strauch and T., 2023+)

Assume $\eta = o(\delta)$ and $\delta^{3/2} = o(\eta)$. Then, for a two-sided Brownian motion $(B^{\leftrightarrow}(h),\ h \in \mathbb{R})$, we have

$$\underbrace{\frac{\eta^2}{\delta^3}}_{=v_\delta^{-1}} \frac{T\|K'\|^2_{L^2}}{2\vartheta^*}(\hat{\tau} - \tau) \xrightarrow{\mathrm{d}} \operatorname*{arg\,min}_{h \in \mathbb{R}} \left\{ B^{\leftrightarrow}(h) + \frac{|h|}{2} \right\}, \quad \text{as } \delta \to 0.$$

The multivariate case



- Recall:

$$\mathrm{d}X(t) = \Delta_\vartheta X(t)\,\mathrm{d}t + \mathrm{d}W(t), \quad \Delta_\vartheta = \nabla \cdot \vartheta \nabla,$$

  with

$$\vartheta(x) = \vartheta_- \mathbf{1}_{\Lambda_-}(x) + \vartheta_+ \mathbf{1}_{\Lambda_+}(x), \quad x \in [0,1]^d = \Lambda_- \uplus \Lambda_+, \vartheta_- \wedge \vartheta_+ > 0.$$

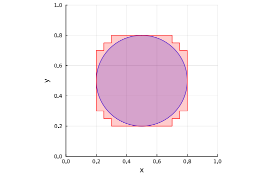
- we call Λ_+ a change domain
- structural similarities to image reconstruction problem

$$Y_i = \vartheta_- \mathbf{1}_{\Lambda_-}(X_i) + \vartheta_+ \mathbf{1}_{\Lambda_+}(X_i) + \varepsilon_i,$$

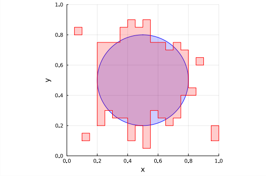
  for (possibly random) measurement locations X_i and noise ε_i

Local observations

- put regular δ-grid on $[0,1]^d$ with grid centers x_α, $\alpha \in [n]^d = [\delta^{-1}]^d$ and aim for estimation of minimal tiling $\Lambda_+^{\updownarrow}$ of Λ_+^0



$\Lambda_+^{\updownarrow}$



$\widehat{\Lambda}_+$

- set $K_{\delta,\alpha} = \delta^{-d/2} K((\cdot - x_\alpha)/\delta)$
- local observations $X_{\delta,\alpha}(t) = \langle X(t), K_{\delta,\alpha}\rangle$ and $X_{\delta,\alpha}^{\Delta}(t) = \langle X(t), \Delta K_{\delta,\alpha}\rangle$ given for $\alpha \in [n]^d$, $t \in [0,T]$

Estimation approach

- $\mathcal{A}_+$ is a family of tiling sets such that $\Lambda_+^{\updownarrow} \in \mathcal{A}_+$
- $\Theta_\pm$ are $\underline{\eta}$-separated sets such that $\vartheta_\pm^0 \in \Theta_\pm$
- set

$$(\hat{\vartheta}_-, \hat{\vartheta}_+, \widehat{\Lambda}_+) \in \operatorname*{arg\,max}_{(\theta_-,\theta_+,\Lambda_+)\in\Theta_-\times\Theta_+\times\mathcal{A}_+} \ell_{\delta,\alpha}(\theta_-, \theta_+, \Lambda_+),$$

where

$$\ell_{\delta,\alpha}(\vartheta_-, \vartheta_+, \Lambda_+) = \vartheta_{\delta,\alpha}(\Lambda_+) \int_0^T X_{\alpha,\delta}^{\Delta}(t)\,\mathrm{d}X_{\delta,\alpha}(t) - \frac{\vartheta_{\delta,\alpha}(\Lambda_+)^2}{2} \int_0^T X_{\delta,\alpha}^{\Delta}(t)^2\,\mathrm{d}t,$$

for

$$\vartheta_{\delta,\alpha}(\Lambda_+) = \begin{cases} \vartheta_+, & x_\alpha \in \Lambda_+, \\ \vartheta_-, & \text{else.} \end{cases}$$

Convergence rate

Theorem (Tiepner and T., 2024+)

Suppose that the number of hypercubes intersecting $\partial\Lambda_+^0$ is of order $\delta^{-d+\beta}$ for some $\beta \in (0,1]$. Then,

$$\mathbb{E}\big[\operatorname{vol}_d(\widehat{\Lambda}_+ \triangle \Lambda_+^0)\big] \lesssim \delta^\beta,$$

and $\hat{\vartheta}_\pm$ are consistent.

In particular, if

- Λ_+^0 (epi)graph of a β-Hölder function $\implies \mathbb{E}\big[\operatorname{vol}_d(\widehat{\Lambda}_+ \triangle \Lambda_+^0)\big] = \mathbb{E}\big[\|\hat{\tau} - \tau^0\|_{L^1}\big] \lesssim \delta^\beta$;
- Λ_+^0 convex $\implies \mathbb{E}\big[\operatorname{vol}_d(\widehat{\Lambda}_+ \triangle \Lambda_+^0)\big] \lesssim \delta$

and in both cases we can choose $\mathcal{A}_+$ s.t. $|\mathcal{A}_+| \asymp \delta^{-d}$.

- in the image reconstruction model with regular design (fixed measurement locations x_α), δ^β is the minimax optimal rate for β-Hölder continuous graph representation of the foreground image
- in the above regular image reconstruction model, δ is optimal for convex foreground images
- optimal rates for higher order Hölder smoothness require randomised design $\rightsquigarrow$ how to incorporate this appropriately in local measurement approach?

Summary

- for a stochastic heat equation with piecewise constant diffusivity, we construct a simultaneous M-estimator for the conductivities $\vartheta^0_\pm$ and the change point τ^0 from multiple local measurements
- in case of non-vanishing jump height, we show that

$$|\hat{\tau} - \tau^0| = \mathcal{O}_{\mathbb{P}}(\delta) \quad \text{and} \quad |\hat{\vartheta}_\pm - \vartheta^0_\pm| = \mathcal{O}_{\mathbb{P}}(\delta^{3/2})$$

- in case of vanishing jump height and known parameters $\vartheta^0_\pm$ we construct a change point estimator $\hat{\tau}$ with asymptotic distribution

$$\frac{\eta^2}{\delta^3} \frac{T\|K'\|^2_{L^2}}{2\vartheta^*}(\hat{\tau} - \tau^0) \xrightarrow{\mathrm{d}} \operatorname*{arg\,min}_{h \in \mathbb{R}} \left\{ B^{\leftrightarrow}(h) + \frac{|h|}{2} \right\}, \qquad \text{as } \delta \to 0,$$

  provided $\eta = o(\delta)$ and $\delta^{3/2} = o(\eta)$
- in the multivariate change domain estimation problem we construct a minimal tiling estimator whose convergence rate is determined by the Minkowski dimension of the change domain's boundary

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Thank you for your attention!