Change point estimation for SPDEs

German Probability and Statistics Days – Essen 2023

Lukas Trottner
joint work with Markus Reiß and Claudia Strauch

08 March 2023

Aarhus University Humboldt-Universität zu Berlin

AARHUS UNIVERSITY

A change point model for a stochastic heat equation

We consider the SPDE

$$\begin{cases} \mathrm{d}X(t) = \Delta_\vartheta X(t)\,\mathrm{d}t + \mathrm{d}W(t), & t \in (0, T], \\ X(0) \equiv 0, & \\ X(t)|_{\{0,1\}} = 0, & t \in (0, T], \end{cases}$$

for space-time white noise $(W(t))_{t\in[0,T]}$ on $L^2((0,1))$ and $\Delta_\vartheta \coloneqq \nabla\vartheta\nabla$, where

$$\vartheta(x) = \vartheta_- \mathbf{1}_{(0,\tau)}(x) + \vartheta_+ \mathbf{1}_{[\tau,1)}(x), \quad x, \tau \in (0,1), 0 < \vartheta_- \wedge \vartheta_+.$$

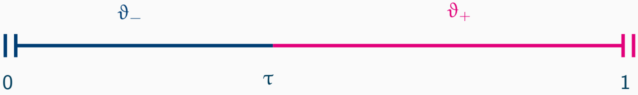
A change point model for a stochastic heat equation

We consider the SPDE

$$\begin{cases} \mathrm{d}X(t) = \Delta_\vartheta X(t)\,\mathrm{d}t + \mathrm{d}W(t), & t \in (0,T], \\ X(0) \equiv 0, & \\ X(t)|_{\{0,1\}} = 0, & t \in (0,T], \end{cases}$$

for space-time white noise $(W(t))_{t\in[0,T]}$ on $L^2((0,1))$ and $\Delta_\vartheta \coloneqq \nabla\vartheta\nabla$, where

$$\vartheta(x) = \vartheta_- \mathbf{1}_{(0,\tau)}(x) + \vartheta_+ \mathbf{1}_{[\tau,1)}(x), \quad x,\tau \in (0,1), 0 < \vartheta_- \wedge \vartheta_+.$$



Positive, self-adjoint operator $-\Delta_\vartheta$ generates a strong analytic semigroup $(S_\vartheta(t) = \exp(t\Delta_\vartheta))_{t\in[0,T]}$, whose transition density obeys the heat kernel bound

$$p_t^\vartheta(x,y) \leqslant c_1 t^{-1/2} \exp\Big(-\frac{|x-y|^2}{c_2 t}\Big), \quad (x,y) \in (0,1)^2, t \in (0,T].$$

$\rightsquigarrow$ $X(t) = \int_0^t S_\vartheta(t-s)\,\mathrm{d}W_s$ is a weak solution, i.e.,

$$\langle X(t), z\rangle = \int_0^t \langle X(s), \Delta_\vartheta z\rangle\,\mathrm{d}s + \langle W(t), z\rangle, \quad z \in D(\Delta_\vartheta) = \big\{u \in H_0^1((0,1)) : \vartheta\nabla u \in H^1((0,1))\big\}.$$

Estimation approach in a related Gaussian white noise model

Consider the SDE

$$\mathrm{d}Y(x) = \vartheta(x)\,\mathrm{d}x + \sigma(x)\,\mathrm{d}B(x), \quad \vartheta(x) = \vartheta_-\mathbf{1}_{(0,\tau^0)}(x) + \vartheta_+\mathbf{1}_{[\tau,1]}(x), x \in [0,1],$$

with known diffusivities $\vartheta_\pm$. Log-likelihood given by

$$\ell(\tau) = \vartheta_- \int_0^\tau \sigma^{-2}(x)\,\mathrm{d}Y(x) - \frac{\vartheta_-^2}{2}\int_0^\tau \sigma^{-2}(x)\,\mathrm{d}x + \vartheta_+ \int_\tau^1 \sigma^{-2}(x)\,\mathrm{d}Y(x) - \frac{\vartheta_+^2}{2}\int_\tau^1 \sigma^{-2}(x)\,\mathrm{d}x,$$

and, for $\eta \coloneqq \vartheta_+ - \vartheta_-$, MLE can be expressed by

$$\widehat{\tau} = \underset{\tau\in[0,1]}{\arg\max}\,\ell(\tau) = \underset{\tau\in[0,1]}{\arg\max}\left\{\int_{\tau\wedge\tau^0}^{\tau\vee\tau^0} \frac{\eta}{\sigma(x)}\,\mathrm{d}B(x) - \frac{1}{2}\int_{\tau\wedge\tau^0}^{\tau\vee\tau^0} \frac{\eta^2}{\sigma(x)^2}\,\mathrm{d}x\right\}.$$

Estimation approach in a related Gaussian white noise model

Consider the SDE

$$\mathrm{d}Y(x) = \vartheta(x)\,\mathrm{d}x + \sigma(x)\,\mathrm{d}B(x), \quad \vartheta(x) = \vartheta_-\mathbf{1}_{(0,\tau^0)}(x) + \vartheta_+\mathbf{1}_{[\tau,1]}(x), x \in [0,1],$$

with known diffusivities $\vartheta_\pm$. Log-likelihood given by

$$\ell(\tau) = \vartheta_- \int_0^\tau \sigma^{-2}(x)\,\mathrm{d}Y(x) - \frac{\vartheta_-^2}{2}\int_0^\tau \sigma^{-2}(x)\,\mathrm{d}x + \vartheta_+ \int_\tau^1 \sigma^{-2}(x)\,\mathrm{d}Y(x) - \frac{\vartheta_+^2}{2}\int_\tau^1 \sigma^{-2}(x)\,\mathrm{d}x,$$

and, for $\eta \coloneqq \vartheta_+ - \vartheta_-$, MLE can be expressed by

$$\widehat{\tau} = \underset{\tau\in[0,1]}{\arg\max}\,\ell(\tau) = \underset{\tau\in[0,1]}{\arg\max}\left\{\int_{\tau\wedge\tau^0}^{\tau\vee\tau^0} \frac{\eta}{\sigma(x)}\,\mathrm{d}B(x) - \frac{1}{2}\int_{\tau\wedge\tau^0}^{\tau\vee\tau^0} \frac{\eta^2}{\sigma(x)^2}\,\mathrm{d}x\right\}.$$

For homoskedastic case $\sigma(x) = n^{-1/2}$ it follows

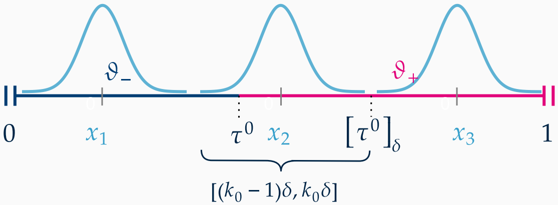
$$\eta^2 n(\widehat{\tau}-\tau^0) \overset{\mathrm{d}}{=} \underset{h\in[-\eta^2 n\tau_0,\eta^2 n(1-\tau_0)]}{\arg\max}\left\{B^{\leftrightarrow}(h)-|h|/2\right\} \xrightarrow{\text{a.s.}} \underset{h\in\mathbb{R}}{\arg\max}\left\{B^{\leftrightarrow}(h)-|h|/2\right\} \overset{\mathrm{d}}{=} \underset{h\in\mathbb{R}}{\arg\min}\left\{B^{\leftrightarrow}(h)+|h|/2\right\},$$

for a two-sided Brownian motion $(B^{\leftrightarrow}(h))_{h\in\mathbb{R}}$, provided $\eta^2 n \to \infty$ as $n \to \infty$ and $\tau^0 \in (0,1)$.

Estimation approach in the SPDE model

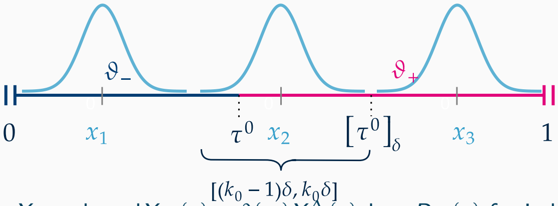
Recall: X mild solution to $\mathrm{d}X(t) = \Delta_\vartheta X(t)\,\mathrm{d}t + \mathrm{d}W(t)$, s.t. Dirichlet boundary conditions

- let $K\colon \mathbb{R} \to \mathbb{R}$ be a smooth kernel with $\operatorname{supp} K \subset [-1/2, 1/2]$, $\|K\|_{L^2} = 1$ and for $\delta = n^{-1}$, $x_i = (i - 1/2)\delta$ ($i \in \{1, \dots, \delta^{-1}\}$), define $K_{\delta,i} = \delta^{-1/2} K(\delta^{-1}(x - x_i))$
- assume that local observations $(X_{\delta,i}(t))_{t\in[0,T]} = (\langle X(t), K_{\delta,i}\rangle)_{t\in[0,T]}$ and $(X^{\Delta}_{\delta,i}(t))_{t\in[0,T]} = (\langle X(t), \Delta K_{\delta,i}\rangle)_{t\in[0,T]}$ are given



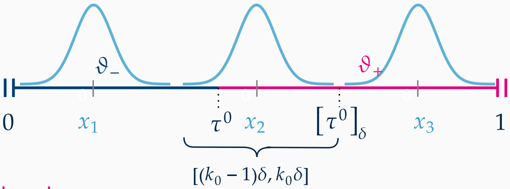
Recall: X mild solution to $\mathrm{d}X(t) = \Delta_\vartheta X(t)\,\mathrm{d}t + \mathrm{d}W(t)$, s.t. Dirichlet boundary conditions

- let $K\colon \mathbb{R} \to \mathbb{R}$ be a smooth kernel with $\operatorname{supp} K \subset [-1/2, 1/2]$, $\|K\|_{L^2} = 1$ and for $\delta = n^{-1}$, $x_i = (i - 1/2)\delta$ ($i \in \{1, \dots, \delta^{-1}\}$), define $K_{\delta,i} = \delta^{-1/2} K(\delta^{-1}(x - x_i))$
- assume that local observations $(X_{\delta,i}(t))_{t\in[0,T]} = (\langle X(t), K_{\delta,i}\rangle)_{t\in[0,T]}$ and $(X^{\Delta}_{\delta,i}(t))_{t\in[0,T]} = (\langle X(t), \Delta K_{\delta,i}\rangle)_{t\in[0,T]}$ are given



- if $|x_i - \tau| \geqslant \delta/2$, then $X_{\delta,i}$ solves $\mathrm{d}X_{\delta,i}(t) = \vartheta(x_i) X^{\Delta}_{\delta,i}(t)\,\mathrm{d}t + B_{\delta,i}(t)$ for independent Brownian motions $(B_{\delta,i}, i = 1, \dots, \delta^{-1})$
- modified local log-likelihood can be expressed by

$$\ell_{\delta,i}(\vartheta_-, \vartheta_+, k) \coloneqq \vartheta_{\delta,i}(k) \int_0^T X^{\Delta}_{\delta,i}(t)\,\mathrm{d}X_{\delta,i}(t) - \frac{\vartheta_{\delta,i}(k)^2}{2} \int_0^T X^{\Delta}_{\delta,i}(t)^2\,\mathrm{d}t, \quad \vartheta_{\delta,i}(k) \coloneqq \begin{cases} \vartheta_-, & i \leqslant k, \\ \vartheta_+, & i > k \end{cases}$$



- modified local log-likelihood:

$$\ell_{\delta,i}(\vartheta_-,\vartheta_+,k) := \vartheta_{\delta,i}(k)\int_0^T X^{\Delta}_{\delta,i}(t)\,\mathrm{d}X_{\delta,i}(t) - \frac{\vartheta_{\delta,i}(k)^2}{2}\int_0^T X^{\Delta}_{\delta,i}(t)^2\,\mathrm{d}t, \quad \vartheta_{\delta,i}(k) := \begin{cases}\vartheta_-, & i\leqslant k,\\ \vartheta_+, & i>k\end{cases}$$

- CUSUM-approach for estimation of $(\vartheta_-^0,\vartheta_+^0,\tau^0)$: $(\widehat{\vartheta}_-^\delta,\widehat{\vartheta}_+^\delta,\widehat{\tau}^\delta) := (\widehat{\vartheta}_-^\delta,\widehat{\vartheta}_+^\delta,\widehat{k}\delta)$, where

$$(\widehat{\vartheta}_-^\delta,\widehat{\vartheta}_+^\delta,\widehat{k}) := \underset{(\vartheta_-,\vartheta_+,k)\in[\underline{\vartheta},\overline{\vartheta}]^2\times[\delta^{-1}]}{\arg\max}\sum_{i\in[\delta^{-1}]}\ell_{\delta,i}(\vartheta_-,\vartheta_+,k)$$

$$= \underset{(\vartheta_-,\vartheta_+,k)\in[\underline{\vartheta},\overline{\vartheta}]^2\times[\delta^{-1}]}{\arg\min}\left\{\frac{1}{2}\sum_{i=1}^{\delta^{-1}}(\vartheta_{\delta,i}(k)-\vartheta^0_{\delta,i})^2 I_{\delta,i} - \sum_{i=1}^{\delta^{-1}}(\vartheta_{\delta,i}(k)-\vartheta^0_{\delta,i})M_{\delta,i} - \vartheta_{\delta,k_0}(k)R_{\delta,k_0}\right\},$$

for

$$M_{\delta,i} := \int_0^T X^{\Delta}_{\delta,i}(t)\,\mathrm{d}B_{\delta,i}(t), \quad I_{\delta,i} := \int_0^T X^{\Delta}_{\delta,i}(t)^2\,\mathrm{d}t,$$

and R_{δ,k_0} is an error term resulting from $K_{\delta,k_0}\notin D(\Delta_\vartheta)$ in general

Lemma [Reiß, Strauch and T. (2023)]

- For any $i \in [\delta^{-1}] \setminus \{k_0\}$,

$$\mathbb{E}[I_{\delta,i}] = \frac{T}{2\vartheta(x_i)} \|K'\|_{L^2}^2 \delta^{-2} + \mathcal{O}(1),$$

  and, moreover, $\mathbb{E}[I_{\delta,k_0}] \sim \delta^{-2}$;
- for any vector $\alpha \in \mathbb{R}^n$ s.t. $\alpha_{k_0} = 0$,

$$\mathrm{Var}\Big(\sum_{i=1}^{\delta^{-1}} \alpha_i I_{\delta,i}\Big) \leqslant \frac{T}{2\underline{\vartheta}^3} \delta^{-2} \|\alpha\|_{\ell^2}^2 \|K'\|_{L^2}^2;$$

- For $\eta \coloneqq \vartheta_+^0 - \vartheta_-^0$,

$$\mathbb{E}[|R_{\delta,k_0}|] \leqslant \frac{\sqrt{3}T}{2\underline{\vartheta}} \|K'\|_{L^2}^2 |\eta| \delta^{-2}.$$

Main observation: $\sum_{i=1}^{\delta^{-1}} \alpha_i (I_{\delta,i} - \mathbb{E}[I_{\delta,i}]$ belongs to second Wiener chaos for an appropriate isonormal Gaussian process associated to $(X_i^{\Delta}(t))_{t\in[0,T],i\in[\delta^{-1}]}$ $\rightsquigarrow$ verify conditions for Bernstein-type concentration inequality of Malliavin differentiable random variables from Nourdin and Viens (2009)[1]

Proposition [Reiß, Strauch and T. (2023)]

Let $\alpha \in \mathbb{R}_+^n \setminus \{0\}$ s.t. $\alpha_{k_0} = 0$. Then, for any $z > 0$, we have

$$\mathbb{P}\Big(\Big|\sum_{i=1}^{n} \alpha_i (I_{\delta,i} - \mathbb{E}[I_{\delta,i}])\Big| \geqslant z\Big) \leqslant 2\exp\Big(-\frac{\underline{\vartheta}^2}{4\|\alpha\|_\infty}\frac{z^2}{z + \sum_{i=1}^{n}\alpha_i \mathbb{E}[I_{\delta,i}]}\Big).$$

[1] I. Nourdin and F.G. Viens (2009). Density formula and concentration inequalities with Malliavin calculus. *Electron. J. Prob.*, 14:no. 78, 2287–2309.

Basic consistency result

Reformulate contrast function in terms of an empiricial process: Let

$$\mathcal{L}_\delta(\vartheta_-,\vartheta_+,h) := \delta^3\Big(\frac{1}{2}\sum_{i=1}^{\delta^{-1}}(\vartheta_{\delta,i}(\lceil h/\delta\rceil)-\vartheta^0_{\delta,i})^2 I_{\delta,i}-\sum_{i=1}^{\delta^{-1}}(\vartheta_{\delta,i}(\lceil h/\delta\rceil)-\vartheta^0_{\delta,i})M_{\delta,i}\Big),\ (\vartheta_-,\vartheta_+,h)\in[\underline{\vartheta},\overline{\vartheta}]^2\times(0,1].$$

Then,

$$\mathcal{L}_\delta(\widehat{\vartheta}^\delta_-,\widehat{\vartheta}^\delta_+,\widehat{\tau}^\delta) = \min_{\chi\in[\underline{\vartheta},\overline{\vartheta}]^2\times(0,1]}\mathcal{L}_\delta(\chi)+\mathcal{O}_{\mathbb{P}}(\delta),$$

and we can show that

$$\sup_{\chi\in[\underline{\vartheta},\overline{\vartheta}]^2\times(0,1]}|\mathcal{L}_\delta(\chi)-\mathbb{E}[\mathcal{L}_\delta(\chi)]| = o_{\mathbb{P}}(1).$$

Theorem [Reiß, Strauch and T. (2023)]

Suppose that $\chi^0(\delta)=(\vartheta^0_-(\delta),\vartheta^0_+(\delta),\tau^0)\xrightarrow[\delta\to 0]{}(\vartheta^*_-,\vartheta^*_+,\tau^0)$. Then, for $\widehat{\chi}^\delta := (\widehat{\vartheta}_-,\widehat{\vartheta}_+,\tau^0)$, it holds $\widehat{\chi}^\delta-\chi^0(\delta)\xrightarrow{\mathbb{P}}0$.

- For the semimetric $\widetilde{d}_\delta$ defined by

$$\widetilde{d}_\delta^2((\vartheta_-,\vartheta_+,h),(\vartheta'_-,\vartheta'_+,h')) \coloneqq |\vartheta_- - \vartheta'_-|^2 + |\vartheta_+ - \vartheta'_+|^2 + |[h]_\delta - [h']_\delta|, \quad [h]_\delta \coloneqq \delta\lceil h/\delta\rceil,$$

  we have the local convexity property

$$\mathbb{E}[\mathcal{L}_\delta(\chi)] - \mathbb{E}[\mathcal{L}_\delta(\chi^0(\delta))] \geqslant c_1 \widetilde{d}_\delta^2(\chi,\chi^0(\delta)), \quad \chi \in B(\chi_\delta^0,\kappa),$$

  for κ small enough, provided that $|\eta| = |\vartheta_+^0(\delta) - \vartheta_-^0(\delta)| \geqslant \underline{\eta} > 0$

Rate of convergence

- For the semimetric $\widetilde{d}_\delta$ defined by

$$\widetilde{d}_\delta^2((\vartheta_-, \vartheta_+, h), (\vartheta'_-, \vartheta'_+, h')) \coloneqq |\vartheta_- - \vartheta'_-|^2 + |\vartheta_+ - \vartheta'_+|^2 + |[h]_\delta - [h']_\delta|, \quad [h]_\delta \coloneqq \delta \lceil h/\delta \rceil,$$

we have the local convexity property

$$\mathbb{E}[\mathcal{L}_\delta(\chi)] - \mathbb{E}[\mathcal{L}_\delta(\chi^0(\delta))] \geqslant c_1 \widetilde{d}_\delta^2(\chi, \chi^0(\delta)), \quad \chi \in B(\chi^0_\delta, \kappa),$$

for κ small enough, provided that $|\eta| = |\vartheta^0_+(\delta) - \vartheta^0_-(\delta)| \geqslant \underline{\eta} > 0$

$\rightsquigarrow$ if we can precisely control the local fluctuations of $\mathcal{L}_\delta(\chi) - \mathbb{E}[\mathcal{L}_\delta(\chi)]$ around $\chi^0(\delta)$, i.e.,

$$\mathbb{E}\Big[\sup_{\widetilde{d}_\delta(\chi, \chi^0(\delta)) < \varepsilon} \big|(\mathcal{L}_\delta - \mathbb{E}[\mathcal{L}_\delta])(\chi) - (\mathcal{L}_\delta - \mathbb{E}[\mathcal{L}_\delta])(\chi^0_\delta)\big|\Big] \leqslant c_2 \psi_\delta(\varepsilon),$$

we can use this information to infer the rate of convergence r_δ by choosing r_δ maximally s.t.

Rate of convergence

- For the semimetric $\widetilde{d}_\delta$ defined by

$$\widetilde{d}_\delta^2((\vartheta_-,\vartheta_+,h),(\vartheta'_-,\vartheta'_+,h')) \coloneqq |\vartheta_- - \vartheta'_-|^2 + |\vartheta_+ - \vartheta'_+|^2 + |[h]_\delta - [h']_\delta|, \quad [h]_\delta \coloneqq \delta\lceil h/\delta\rceil,$$

  we have the local convexity property

$$\mathbb{E}[\mathcal{L}_\delta(\chi)] - \mathbb{E}[\mathcal{L}_\delta(\chi^0(\delta))] \geqslant c_1 \widetilde{d}_\delta^2(\chi,\chi^0(\delta)), \quad \chi \in B(\chi_\delta^0,\kappa),$$

  for κ small enough, provided that $|\eta| = |\vartheta_+^0(\delta) - \vartheta_-^0(\delta)| \geqslant \underline{\eta} > 0$

⇝ if we can precisely control the local fluctuations of $\mathcal{L}_\delta(\chi) - \mathbb{E}[\mathcal{L}_\delta(\chi)]$ around $\chi^0(\delta)$, i.e.,

$$\mathbb{E}\Big[\sup_{\widetilde{d}_\delta(\chi,\chi^0(\delta))<\varepsilon} \big|(\mathcal{L}_\delta - \mathbb{E}[\mathcal{L}_\delta])(\chi) - (\mathcal{L}_\delta - \mathbb{E}[\mathcal{L}_\delta])(\chi_\delta^0)\big|\Big] \leqslant c_2\psi_\delta(\varepsilon),$$

  we can use this information to infer the rate of convergence r_δ by choosing r_δ maximally s.t.

  (i) $\delta \mapsto r_\delta^2\psi_\delta(r_\delta^{-1})$ is bounded

Rate of convergence

- For the semimetric $\widetilde{d}_\delta$ defined by

$$\widetilde{d}_\delta^2((\vartheta_-,\vartheta_+,h),(\vartheta'_-,\vartheta'_+,h')) := |\vartheta_- - \vartheta'_-|^2 + |\vartheta_+ - \vartheta'_+|^2 + |[h]_\delta - [h']_\delta|, \quad [h]_\delta := \delta\lceil h/\delta\rceil,$$

we have the local convexity property

$$\mathbb{E}[\mathcal{L}_\delta(\chi)] - \mathbb{E}[\mathcal{L}_\delta(\chi^0(\delta))] \geqslant c_1 \widetilde{d}_\delta^2(\chi,\chi^0(\delta)), \quad \chi \in B(\chi^0_\delta,\kappa),$$

for κ small enough, provided that $|\eta| = |\vartheta^0_+(\delta) - \vartheta^0_-(\delta)| \geqslant \underline{\eta} > 0$

$\rightsquigarrow$ if we can precisely control the local fluctuations of $\mathcal{L}_\delta(\chi) - \mathbb{E}[\mathcal{L}_\delta(\chi)]$ around $\chi^0(\delta)$, i.e.,

$$\mathbb{E}\Big[\sup_{\widetilde{d}_\delta(\chi,\chi^0(\delta))<\varepsilon} \Big|(\mathcal{L}_\delta - \mathbb{E}[\mathcal{L}_\delta])(\chi) - (\mathcal{L}_\delta - \mathbb{E}[\mathcal{L}_\delta])(\chi^0_\delta)\Big|\Big] \leqslant c_2\psi_\delta(\varepsilon),$$

we can use this information to infer the rate of convergence r_δ by choosing r_δ maximally s.t.

(i) $\delta \mapsto r_\delta^2\psi_\delta(r_\delta^{-1})$ is bounded
(ii) $\mathcal{L}_\delta(\widehat{\chi}^\delta) \leqslant \inf_{\chi\in[\underline{\vartheta},\overline{\vartheta}]^2\times(0,1]} \mathcal{L}_\delta(\chi) + \mathcal{O}_{\mathbb{P}}(r_\delta^{-2})$

Rate of convergence

- For the semimetric $\widetilde{d}_\delta$ defined by
$$\widetilde{d}_\delta^2((\vartheta_-,\vartheta_+,h),(\vartheta'_-,\vartheta'_+,h')) := |\vartheta_- - \vartheta'_-|^2 + |\vartheta_+ - \vartheta'_+|^2 + |[h]_\delta - [h']_\delta|, \quad [h]_\delta := \delta\lceil h/\delta\rceil,$$
we have the local convexity property
$$\mathbb{E}[\mathcal{L}_\delta(\chi)] - \mathbb{E}[\mathcal{L}_\delta(\chi^0(\delta))] \geqslant c_1 \widetilde{d}_\delta^2(\chi,\chi^0(\delta)), \quad \chi \in B(\chi_\delta^0,\kappa),$$
for κ small enough, provided that $|\eta| = |\vartheta_+^0(\delta) - \vartheta_-^0(\delta)| \geqslant \underline{\eta} > 0$

⇝ if we can precisely control the local fluctuations of $\mathcal{L}_\delta(\chi) - \mathbb{E}[\mathcal{L}_\delta(\chi)]$ around $\chi^0(\delta)$, i.e.,
$$\mathbb{E}\Big[\sup_{\widetilde{d}_\delta(\chi,\chi^0(\delta))<\varepsilon} \Big|(\mathcal{L}_\delta - \mathbb{E}[\mathcal{L}_\delta])(\chi) - (\mathcal{L}_\delta - \mathbb{E}[\mathcal{L}_\delta])(\chi_\delta^0)\Big|\Big] \leqslant c_2\psi_\delta(\varepsilon),$$
we can use this information to infer the rate of convergence r_δ by choosing r_δ maximally s.t.
  (i) $\delta \mapsto r_\delta^2\psi_\delta(r_\delta^{-1})$ is bounded
  (ii) $\mathcal{L}_\delta(\widehat{\chi}^\delta) \leqslant \inf_{\chi\in[\underline{\vartheta},\overline{\vartheta}]^2\times(0,1]} \mathcal{L}_\delta(\chi) + \mathcal{O}_{\mathbb{P}}(r_\delta^{-2})$

Theorem [Reiß, Strauch and T. (2023)]

Suppose that $\chi^0(\delta) \underset{\delta\to 0}{\longrightarrow} \chi^*$ and that $|\eta| \geqslant \underline{\eta}$ for all $\delta \in 1/\mathbb{N}$. Then, $\delta^{-1/2}\widetilde{d}_\delta(\widehat{\chi}_\delta,\chi^0(\delta)) = \mathcal{O}_{\mathbb{P}}(1)$. In particular,
$$\widehat{\tau}^\delta - \tau^0 = \mathcal{O}_{\mathbb{P}}(\delta) \quad \text{and} \quad \widehat{\vartheta}_\pm - \vartheta_\pm^0 = \mathcal{O}_{\mathbb{P}}(\delta^{1/2}).$$

Limit theorem for vanishing jump height

- for the previous consistency result it was crucial that the jump height η does not vanish
- assume now that $\eta \underset{\delta \to 0}{\longrightarrow} 0$ and that the nuisance parameters $\vartheta_\pm^0 = \vartheta_\pm^0(\delta)$ are known
- CUSUM estimator: $\widehat{\tau} = \widehat{k}\delta$, where

$$\begin{aligned}\widehat{k} \coloneqq \underset{k=1,\dots,\delta^{-1}}{\arg\max} \sum_{i=1}^{k} &\left(\vartheta_-^0 \int_0^T X_{\delta,i}^{\Delta}(t)\,\mathrm{d}X_{\delta,i}(t) - \frac{(\vartheta_-^0)^2}{2} \int_0^T X_{\delta,i}^{\Delta}(t)^2\,\mathrm{d}t\right) \\ &+ \sum_{i=k+1}^{\delta^{-1}} \left(\vartheta_+^0 \int_0^T X_{\delta,i}^{\Delta}(t)\,\mathrm{d}X_{\delta,i}(t) - \frac{(\vartheta_+^0)^2}{2} \int_0^T X_{\delta,i}^{\Delta}(t)^2\,\mathrm{d}t\right) \\ &= \underset{k=1,\dots,\delta^{-1}}{\arg\max}\, Z_k,\end{aligned}$$

for

$$Z_k = \begin{cases} 0, & k = k_0, \\ \eta \sum_{i=k+1}^{k_0} \int_0^T X_{\delta,i}^{\Delta}(t)\,\mathrm{d}B_{\delta,i}(t) - \frac{\eta^2}{2} \sum_{i=k+1}^{k_0} \int_0^T X_{\delta,i}^{\Delta}(t)^2\,\mathrm{d}t + \eta R_{\delta,k_0}, & k < k_0, \\ -\eta \sum_{i=k_0+1}^{k} \int_0^T X_{\delta,i}^{\Delta}(t)\,\mathrm{d}B_{\delta,i}(t) - \frac{\eta^2}{2} \sum_{i=k_0+1}^{k} \int_0^T X_{\delta,i}^{\Delta}(t)^2\,\mathrm{d}t, & k > k_0, \end{cases}.$$

Limit theorem for vanishing jump height

Reformulate the estimator again in terms of an M-estimator: Let $v_\delta = \delta^3/\eta^2$, and define $M^{\tau^0}_{T,\delta}(h)$, $I^{\tau^0}_{T,\delta}(h)$ appropriately s.t. for

$$\mathcal{Z}_\delta(h) \coloneqq \eta M^{\tau^0}_{T,\delta}(h) + \frac{\eta^2}{2} I^{\tau^0}_{T,\delta}(h),$$

we have

$$\mathcal{Z}_\delta(v_\delta^{-1}(\widehat{\tau} - \tau^0)) = \min_{h\in\mathbb{R}} \mathcal{Z}_\delta(h) + \mathcal{O}_{\mathbb{P}}(\eta^2\delta^{-2}) = \min_{h\in\mathbb{R}} \mathcal{Z}_\delta(h) + o_{\mathbb{P}}(1),$$

provided $\eta = o(\delta)$.

Limit theorem for vanishing jump height

Reformulate the estimator again in terms of an M-estimator: Let $v_\delta = \delta^3/\eta^2$, and define $M^{\tau^0}_{T,\delta}(h), I^{\tau^0}_{T,\delta}(h)$ appropriately s.t. for

$$\mathcal{Z}_\delta(h) := \eta M^{\tau^0}_{T,\delta}(h) + \tfrac{\eta^2}{2} I^{\tau^0}_{T,\delta}(h),$$

we have

$$\mathcal{Z}_\delta(v_\delta^{-1}(\widehat{\tau} - \tau^0)) = \min_{h \in \mathbb{R}} \mathcal{Z}_\delta(h) + \mathcal{O}_{\mathbb{P}}(\eta^2\delta^{-2}) = \min_{h \in \mathbb{R}} \mathcal{Z}_\delta(h) + o_{\mathbb{P}}(1),$$

provided $\eta = o(\delta)$.

Theorem [Reiß, Strauch and T. (2023)]

Assume $\eta = o(\delta)$ and $\delta^{3/2} = o(\eta)$. Then, for a two-sided Brownian motion $(B^{\leftrightarrow}(h), h \in \mathbb{R})$, we have

$$v_\delta^{-1} \frac{T\|K'\|^2_{L^2}}{2\vartheta^*}(\widehat{\tau} - \tau) \xrightarrow{\mathrm{d}} \operatorname*{arg\,min}_{h \in \mathbb{R}} \Big\{ B^{\leftrightarrow}(h) + \frac{|h|}{2} \Big\}, \qquad \text{as } \delta \to 0.$$

Summary

- for a stochastic heat equation with piecewise constant diffusivity, we construct a simultaneous M-estimator for the nuisance parameters $\vartheta_\pm^0$ and the change point τ^0 from multiple local measurements
- we prove consistency of the estimator and, in case of non-vanishing jump height, demonstrate

$$\widehat{\tau}^\delta - \tau^0 = \mathcal{O}_{\mathbb{P}}(\delta) \quad \text{and} \quad \widehat{\vartheta}_\pm - \vartheta_\pm^0 = \mathcal{O}_{\mathbb{P}}(\delta^{1/2}))$$

- in case of vanishing jump height and known nuisance parameters $\vartheta_\pm^0$ we construct a change point estimator $\widehat{\tau}$ obeying the limit theorem

$$\frac{\eta^2}{\delta^3}\frac{T\|K'\|_{L^2}^2}{2\vartheta^*}(\widehat{\tau}-\tau^0) \xrightarrow{\mathrm{d}} \operatorname*{arg\,min}_{h\in\mathbb{R}}\left\{B^{\leftrightarrow}(h)+\frac{|h|}{2}\right\}, \qquad \text{as } \delta\to 0,$$

provided $\eta = o(\delta)$ and $\delta^{3/2} = o(\eta)$

Summary

- for a stochastic heat equation with piecewise constant diffusivity, we construct a simultaneous M-estimator for the nuisance parameters $\vartheta^0_\pm$ and the change point τ^0 from multiple local measurements
- we prove consistency of the estimator and, in case of non-vanishing jump height, demonstrate

$$\widehat{\tau}^\delta - \tau^0 = \mathcal{O}_{\mathbb{P}}(\delta) \quad \text{and} \quad \widehat{\vartheta}_\pm - \vartheta^0_\pm = \mathcal{O}_{\mathbb{P}}(\delta^{1/2}))$$

- in case of vanishing jump height and known nuisance parameters $\vartheta^0_\pm$ we construct a change point estimator $\widehat{\tau}$ obeying the limit theorem

$$\frac{\eta^2}{\delta^3}\frac{T\|K'\|^2_{L^2}}{2\vartheta^*}(\widehat{\tau}-\tau^0) \xrightarrow{\mathrm{d}} \operatorname*{arg\,min}_{h\in\mathbb{R}}\left\{B^{\leftrightarrow}(h)+\frac{|h|}{2}\right\}, \qquad \text{as } \delta \to 0,$$

provided $\eta = o(\delta)$ and $\delta^{3/2} = o(\eta)$

Thank you for your attention!