Mathematical Foundations of SPDEs

Pathways into Mathematics of SPDEs: A Workshop for Young Researchers

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Overview

Literature

- Da Prato, G., Zabczyk, J., *Stochastic Equations in Infinite Dimensions.* 2nd ed., Cambridge University Press, 2014.
- Hairer, M., *An Introduction to Stochastic PDEs*, 2023. https://www.hairer.org/notes/SPDEs.pdf
- Liu, W., Röckner, M., *Stochastic Partial Differential Equations: An Introduction*, 1st ed., Springer Cham, 2025.
- Rohde, A., *Stochastische partielle Differentialgleichungen*, Lecture Notes University of Freiburg, 2023.

PDE basics

(Deterministic) Heat equation

Let $\mathcal{O}$ be a domain in $\mathbb{R}^d$.

Heat equation

The heat equation with Dirichlet boundary conditions and initial condition $g \in L^2(\mathcal{O})$:

$$\begin{cases} \dot{u}(t,x) = \Delta u(t,x), & t \in (0,T], x \in \mathcal{O}, \\ u(t,x) = 0, & t \in (0,T], x \in \partial\mathcal{O}, \\ u(0,x) = g(x), & x \in \mathcal{O}, \end{cases}$$

where $\Delta u(t,x) := \sum_{i=1}^{d} \partial_{x_i}^2 u(t,x)$ and $\dot{u}(t,x) := \partial_t u(t,x)$. The associated integral equation is $u(t,x) = g(x) + \int_0^t \Delta u(s,x)\,\mathrm{d}s$.

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Requirements on a strong solution in an L^2-sense:

- $\dot{u} = \partial_t u \in L^2(0,T; L^2(\mathcal{O}))$ exists in a weak sense.
- for a.e. $t \in (0,T]$, $\Delta u(t,\cdot) \in L^2(\mathcal{O})$
  $\rightsquigarrow u \in L^2(0,T; H^2(\mathcal{O}))$
- u vanishes in an appropriate sense on the boundary $\rightsquigarrow u(t,\cdot) \in L^2(0,T; H_0^1(\mathcal{O}))$

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$(u(t))_{t\in[0,T]} := (u(t,\cdot))_{t\in[0,T]}$ is called a weak solution if

- $t \mapsto u(t,\cdot) \in \mathcal{C}(0,T; L^2(\mathcal{O}))$.
- for any test function $\varphi \in D(\Delta) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$:

$$\langle u(t), \varphi \rangle = \langle g, \varphi \rangle + \int_0^t \langle u(s), \Delta\varphi \rangle \,\mathrm{d}s, \quad \forall t \in [0,T].$$

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To keep the presentation light, we will mostly focus on weak solutions in the following.

Solving the heat equation

$\mathcal{O} = \mathbb{R}^d$: define

- the heat kernel $K_t(x, y) := \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}}$;
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$$S(t) : \begin{cases} L^2(\mathbb{R}^d) \mapsto L^2(\mathbb{R}^d), \\ z \mapsto S(t)z := \int_{\mathbb{R}^d} K_t(\cdot, y) z(y) \, \mathrm{d}y, \end{cases}$$

  for $t > 0$ and $S(0) = I$

Then, for any intial condition $u(0) = g \in L^2(\mathbb{R}^d)$, the mild solution

$$u(t) = S(t)g$$

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- the Dirichlet Laplacian Δ is a self-adjoint operator with discrete spectrum consisting of eigenvalues $0 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots$
- define $S(0) = I$ and for $t > 0$, $S(t) = \mathrm{e}^{t\Delta}$ via spectral calculus: for orthornormal eigenpairs $(\lambda_k, e_k)_{k \geq 1}$ of Δ set

$$S(t)z := \sum_{k \in \mathbb{N}} \mathrm{e}^{\lambda_k t} \langle e_k, z \rangle e_k.$$

For any intial condition $u(0) = g \in L^2(\mathbb{R}^d)$, the unique weak solution is given by the mild solution

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In both cases, $(S(t))_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on $L^2(\mathcal{O})$ in the sense

- $S(t+s) = S(t)S(s), \quad \forall s, t \geq 0;$
- $\lim_{t\downarrow 0}\|S(t)z - z\| = 0, \quad \forall z \in L^2(\mathcal{O})$

Moreover, for any $z \in L^2(\mathcal{O})$: $\frac{\mathrm{d}}{\mathrm{d}t} S(t)z = \Delta S(t)z \quad [= S(t)\Delta z \text{ if } z \in D(\Delta)]$

The inhomogeneous heat equation

Let us now introduce an external heat source $f(t,x)$ to the heat equation, i.e., consider the initial value problem

$$\begin{cases} \dot{u}(t,x) = \Delta u(t,x) + f(t,x), & t \in (0,T], x \in \mathcal{O}, \\ u(t,x) = 0, & t \in (0,T], x \in \partial\mathcal{O}, \\ u(0,x) = g(x), & x \in \mathcal{O}. \end{cases}$$

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Assume $g \in L^2(\mathcal{O})$ and $\int_0^T \|f(t,\cdot)\|_{L^2} \, \mathrm{d}t < \infty$ and define the mild solution via the variation of constants formula

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Then, for any $\varphi \in D(\Delta) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$:

$$\begin{aligned} \int_0^t \langle u(s), \Delta\varphi \rangle\, \mathrm{d}s &= \int_0^t \langle S(s)g, \Delta\varphi \rangle\, \mathrm{d}s + \int_0^t \int_0^s \langle S(s-v)f(v), \Delta\varphi \rangle\, \mathrm{d}v\, \mathrm{d}s \\ &= \int_0^t \langle g, S(s)\Delta\varphi \rangle\, \mathrm{d}s + \int_0^t \int_0^s \langle f(v), S(s-v)\Delta\varphi \rangle\, \mathrm{d}v\, \mathrm{d}s \\ &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \langle g, S(s)\varphi \rangle\, \mathrm{d}s + \int_0^t \int_v^t \langle f(v), S(s-v)\Delta\varphi \rangle\, \mathrm{d}s\, \mathrm{d}v \\ &= \langle g, (S(t)-I)\varphi \rangle + \int_0^t \int_0^{t-v} \langle f(v), \Delta S(w)\varphi \rangle\, \mathrm{d}w\, \mathrm{d}v \end{aligned}$$

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$$\begin{aligned} &= \langle (S(t)-I)g, \varphi\rangle + \int_0^t \int_0^{t-v} \frac{\mathrm{d}}{\mathrm{d}w}\langle f(v), S(w)\varphi\rangle\,\mathrm{d}w\,\mathrm{d}v \\ &= \langle (S(t)-I)g, \varphi\rangle + \int_0^t \langle S(t-v)f(v), \varphi\rangle\,\mathrm{d}v - \int_0^t \langle f(v), \varphi\rangle\,\mathrm{d}v \\ &= \langle u(t), \varphi\rangle - \langle g, \varphi\rangle - \int_0^t \langle f(s), \varphi\rangle\,\mathrm{d}s \end{aligned}$$

$$\iff \langle u(t), \varphi\rangle = \langle g, \varphi\rangle + \int_0^t \langle u(s), \Delta\varphi\rangle\,\mathrm{d}s + \int_0^t \langle f(s), \varphi\rangle\,\mathrm{d}s$$

$\Longrightarrow$ $(u(t))_{t\in[0,T]}$ is a weak solution

The general Cauchy problem for nonhomogeneous equations

We replace the Dirichlet-Laplacian Δ on $L^2(\mathcal{O})$ with the infinitesimal generator A of a C_0-semigroup $(S(t))_{t\geq 0}$ on a separable Hilbert space $\mathcal{H}$, i.e., A is a linear operator given by

$$\begin{cases} D(A) = \left\{z \in \mathcal{H} \;:\; \mathcal{H}\text{-}\lim_{t\downarrow 0} \frac{S(t)z - z}{t} \text{ exists}\right\} \\ D(A) \ni z \mapsto Az := \mathcal{H}\text{-}\lim_{t\downarrow 0} \frac{S(t)z - z}{t}. \end{cases}$$

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For $g \in \mathcal{H}$ and $f \in L^1([0,T];\mathcal{H})$, the mild solution $u(t) = S(t)g + \int_0^t S(t-s)f(s)\,\mathrm{d}s$ is the unique weak solution for the Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t) + f(t), \quad t \in (0,T], \\ u(0) = g, \end{cases}$$

in the sense

$$\forall \varphi \in D(A^*), t \in [0,T] \,:\, \langle u(t), \varphi \rangle = \langle g, \varphi \rangle + \int_0^t \big(\langle u(s), A^*\varphi \rangle + \langle f(s), \varphi \rangle \big)\,\mathrm{d}s.$$

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- Hille–Yosida theorem: full characterisation of generators of C_0-semigroups satisfying $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. In particular, $D(A)$ needs to be dense in $\mathcal{H}$ and the resolvent $R_\lambda = \int_0^\infty \mathrm{e}^{-\lambda t} S(t) \cdot \mathrm{d}t$ needs to exist as a bounded linear operator for all $\lambda > \omega$.
- analytic semigroups are particularly nice to work with since they have the smoothing effect $S(t)z \in D(A)$ for all $t > 0$ and $z \in \mathcal{H}$

Examples of generators

- any bounded linear operator A on $\mathcal{H}$ generates the semigroup $S(t) = \mathrm{e}^{tA} := \sum_{n \in \mathbb{N}_0} \frac{t^n A^n}{n!}$.
- elliptic diffusion generators

$$Az(x) = \sum_{i,j=1}^{n} (\sigma\sigma^\top)_{i,j}(x) \frac{\partial^2}{\partial_{x_i}\partial_{x_j}} z(x) + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} z(x) = \mathrm{Tr}(\sigma^\top(x)\nabla^2 z(x)\sigma(x)) + b(x)^\top \nabla z(x),$$

  where $|\sigma^\top(x)y|^2 \geq \underline{\lambda} > 0$ for all $x, y \in \mathcal{O}$ and σ, b are continuous on $\overline{\mathcal{O}}$.
- divergence form operators $A = \nabla \cdot \vartheta \nabla$ for a measurable diffusivity $\vartheta \colon \mathcal{O} \to [\underline{\vartheta}, \overline{\vartheta}] \subset (0, \infty)$

Next, we wish to draw connections between stochastic differential equations (SDEs) and PDEs by discussing probabilistic characterisations of solutions of Cauchy problems associated to elliptic diffusion generators.

PDE solutions from the perspective of stochastic analysis

The finite-dimensional stochastic integral

- let $(B(t))_{t\geq 0}$ be a $\mathbb{F}$-Brownian motion for a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, i.e.,
  - $B_0 = 0$ a.s.;
  - $t \mapsto B_t$ is continuous a.s.;
  - for all $0 \leq s \leq t$, $B(t) - B(s) \overset{d}{=} B(t-s) \sim \mathcal{N}(0, t-s)$;
  - for all $0 \leq s < t$, $B(t) - B(s)$ is independent of $\mathcal{F}_s$.
- we want to define a stochastic integral $\int_0^t \Phi(s)\,\mathrm{d}B(s)$ for appropriate stochastic processes Φ
- this cannot be defined ω-wise as a Stieltjes integral because B has unbounded variation:

$$\lim_{\|\Pi(t)\|\to 0} \sum_{s\in\Pi(t)\setminus\{t\}} |B(s') - B(s)| = \infty, \quad \mathbb{P}\text{-a.s.},$$

  where $\Pi(t)$ is a finite partition of $[0, t]$ with mesh $\|\Pi(t)\|$ and $s' = \min\{u \in \Pi(t) \,:\, u > s\}$ for $s \in \Pi(t) \setminus \{t\}$.
- however, the quadratic variation is finite:

$$\langle\!\langle B \rangle\!\rangle_t := L^2\text{-}\lim_{\|\Pi(t)\|\to 0} \sum_{s\in\Pi(t)\setminus\{t\}} |B(s') - B(s)|^2 = t$$

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The general construction scheme is as follows:

1. define the class $\mathcal{E}_T$ of elementary processes on $[0, T]$ as processes of the form

$$\Phi(t) = \sum_{s\in\Pi(T)\setminus\{T\}} \Psi_s \mathbf{1}_{(s,s']}(t), \quad t \in [0, T]$$

   for some partition $\Pi(T)$ and random variables $\Psi_s \in \mathcal{F}_s$ and let $\mathcal{M}_T^2$ be the space of càdlàg square-integrable $\mathbb{F}$-martingales[a] on $[0, T]$
2. define the stochastic integral $I(\Phi) = \int_0^\cdot \Phi(s)\,\mathrm{d}B(s)$ for elementary processes via the mapping

$$I : \mathcal{E}_T \to \mathcal{M}_T^2, \; I(\Phi)(t) = \sum_{s\in\Pi(T)} \Psi_s(B(s'\wedge t) - B(s\wedge t)).$$

   Since B is a continuous martingale with quadratic variation $\langle\!\langle B \rangle\!\rangle_t = t$ we obtain

$$\|I(\Phi)\|_{\mathcal{M}_T^2}^2 := \mathbb{E}\big[(I(\Phi)(T))^2\big] = \mathbb{E}\Big[\int_0^T |\Phi(t)|^2\,\mathrm{d}t\Big] =: \|\Phi\|_T^2,$$

   so I is an isometry.

[a] a process $(M_t)_{t\in[0,T]}$ is called an $\mathbb{F}$-martingale if $M_t \in L^1(\mathbb{P})$ and $\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$ for all $0 \leq s \leq t \leq T$

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   for some partition $\Pi(T)$ and random variables $\Psi_s \in \mathcal{F}_s$ and let $\mathcal{M}^2_T$ be the space of càdlàg square-integrable $\mathbb{F}$-martingales[a] on $[0,T]$

2. define the stochastic integral $I(\Phi) = \int_0^\cdot \Phi(s)\,\mathrm{d}B(s)$ for elementary processes via the mapping

$$I : \mathcal{E}_T \to \mathcal{M}^2_T,\ I(\Phi)(t) = \sum_{s\in\Pi(T)} \Psi_s(B(s'\wedge t) - B(s\wedge t)).$$

   Since B is a continuous martingale with quadratic variation $\langle\!\langle B\rangle\!\rangle_t = t$ we obtain

$$\|I(\Phi)\|^2_{\mathcal{M}^2_T} := \mathbb{E}\big[(I(\Phi)(T))^2\big] = \mathbb{E}\Big[\int_0^T |\Phi(t)|^2\,\mathrm{d}t\Big] =: \|\Phi\|^2_T,$$

   so I is an isometry.

3. Since $\mathcal{M}^2_T$ is a Banach space and I is an isometry, we can extend the stochastic integral to processes in the closure $\overline{\mathcal{E}}_T := \overline{\mathcal{E}_T}^{\|\cdot\|_T}$. The Itô-isometry

$$\mathbb{E}\Big[\Big(\int_0^T \Phi(t)\,\mathrm{d}B(t)\Big)^2\Big] = \mathbb{E}\Big[\int_0^T |\Phi(t)|^2\,\mathrm{d}t\Big]$$

   extends to all $\Phi\in\overline{\mathcal{E}}_T$.

4. $\overline{\mathcal{E}}_T$ can be characterised as the class of processes Φ such that $\|\Phi\|_T < \infty$ and which are $\mathbb{F}$-predictable, i.e., processes that are measurable w.r.t.

$$\sigma\big(Y : \Omega\times[0,T]\to\mathbb{R} :\ Y \text{ is left-continuous and } \mathbb{F}\text{-adapted}\big)$$

5. if $B = (B_1,\dots,B_d)$ is a d-dimensional Brownian motion and $\Phi : \Omega\times[0,T]\to\mathbb{R}^{m\times d}$ is a matrix-valued predictable process, we define

$$\int_0^t \Phi(s)\,\mathrm{d}B(s) = \Big(\sum_{j=1}^d \int_0^t \Phi_{i,j}(s)\,\mathrm{d}B_j(s)\Big)^\top_{i=1,\dots,m}$$
$$= \sum_{i=1}^m\sum_{j=1}^d \int_0^t \langle \Phi(s)e_j, e_i\rangle\,\mathrm{d}B_j(s)e_i.$$

[a] a process $(M_t)_{t\in[0,T]}$ is called an $\mathbb{F}$-martingale if $M_t\in L^1(\mathbb{P})$ and $\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$ for all $0\le s\le t\le T$

From SDEs to PDEs

Consider the stochastic differential equation on $\mathbb{R}^d$, given by

$$\mathrm{d}X(t) = b(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}B(t), \quad X_0 = \xi.$$

for a Lipschitz drift $b : \mathbb{R}^d \to \mathbb{R}^d$ and a uniformly elliptic, bounded continuous matrix function $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$.
Under these conditions there exists a unique strong solution: for a fixed $\mathbb{F}$-Brownian motion, this is an $\mathbb{F}$-predictable process X such that almost surely,

$$X(t) = \xi + \int_0^t b(X(s))\,\mathrm{d}s + \int_0^t \sigma(X(s))\,\mathrm{d}B(s), \quad t \geq 0.$$

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- let X^x denote the strong solution given $X(0) = x$.
- fix a bounded domain $\mathcal{O}$ and let $\tau^x_{\mathcal{O}} = \inf\{t \geq 0 : X^x_t \in \partial O\}$. On the space of bounded measurable functions $\mathcal{B}_b(\mathcal{O})$ define the sub-Markov C_0-semigroup $(P(t))_{t\geq 0}$ by

  $$P(t)f(x) = \mathbb{E}\big[f(X^x_t)\mathbf{1}_{\{t<\tau^x_{\mathcal{O}}\}}\big], \quad x \in \mathcal{O}, t \geq 0,$$

  which is the transition semigroup of the diffusion killed when exiting $\mathcal{O}$
- it can be shown that this semigroup has the Feller property $P(t)\mathcal{C}_0(\mathcal{O}) \subset \mathcal{C}_0(\mathcal{O})$
- $(P(t))_{t\geq 0}$ can be extended to a C_0-semigroup $(S(t))_{t\geq 0}$ on $L^2(\mathcal{O})$.

From SDEs to PDEs

- Itô's formula gives for any $\varphi \in \mathcal{C}_c^2(\mathcal{O})$ and $A\varphi(x) = \frac{1}{2}\sum_{i,j=1}^n (\sigma\sigma^\top)_{i,j}(x)\frac{\partial^2}{\partial_{x_i}\partial_{x_j}}\varphi(x) + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i}\varphi(x)$,

$$\varphi(X^x(t))\mathbf{1}_{\{t<\tau_{\mathcal{O}}^x\}} = \Big(\varphi(x) + \int_0^t A\varphi(X^x(s))\,\mathrm{d}s + \int_0^t \nabla\varphi(X^x(s))\cdot\sigma(X^x(s))\,\mathrm{d}B(s)\Big)\mathbf{1}_{\{t<\tau_{\mathcal{O}}^x\}}$$

$$= \Big(\varphi(x) + \int_0^{t\wedge\tau_{\mathcal{O}}^x} A\varphi(X^x(s))\,\mathrm{d}s + \int_0^{t\wedge\tau_{\mathcal{O}}^x} \nabla\varphi(X^x(s))\cdot\sigma(X^x(s))\,\mathrm{d}B(s)\Big)\mathbf{1}_{\{t<\tau_{\mathcal{O}}^x\}}$$

$$\Longrightarrow \quad P(t)\varphi(x) = \mathbb{E}\big[\varphi(X^x(t))\mathbf{1}_{\{t<\tau_{\mathcal{O}}^x\}}\big] = \varphi(x) + \int_0^t \mathbb{E}\big[A\varphi(X^x(s))\mathbf{1}_{\{s<\tau_{\mathcal{O}}^x\}}\big]\,\mathrm{d}s - \mathbb{E}\big[\varphi(X^x(t))\mathbf{1}_{\{\tau_{\mathcal{O}}^x\le t\}}\big]$$

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- since $(P(t))_{t\ge 0}$ is Feller, the limit holds uniformly in x; boundedness of $\mathcal{O}$ thus yields

$$L^2\text{-}\lim_{t\downarrow 0}\frac{S(t)\varphi-\varphi}{t} = L^2\text{-}\lim_{t\downarrow 0}\frac{P(t)\varphi-\varphi}{t} = A\varphi.$$

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- $(x,t)\mapsto\mathbb{E}[\varphi(X_t^x)]$ solves the Kolmogorov backward equation $\begin{cases}\dot u(t,x) = Au(t,x), & t\in(0,T], x\in\mathcal{O},\\ u(t,x)=0, & t\in(0,T], x\in\partial\mathcal{O},\\ u(0,x)=\varphi(x), & x\in\mathcal{O}.\end{cases}$

Infinite dimensional stochastic analysis and SPDEs

Stochastic heat equation (informal)

Recall that if $(S_t)_{t\geq 0} = (\mathrm{e}^{t\Delta})_{t\geq 0}$ is the C_0-semigroup generated by the Dirichlet Laplacian on $L^2(\mathcal{O})$, then $u(t) = S(t)g + \int_0^t S(t-s)f(s)\,\mathrm{d}s$ is the unique weak solution of the heat equation

$$\begin{cases} \dot{u}(t) = \Delta u(t) + f(t), & t \in (0,T], \\ u(0) = g, \end{cases}$$

provided the external heat source $f : [0,T] \to L^2(\mathcal{O})$ is integrable.

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Stochastic heat equation (idea): replace f with (or add to f) some Gaussian random function $\dot{W}^Q(t,x)$ such that

(i) q is a generalised function on $\mathcal{O}$ and the symmetric nonnegative covariance operator Q is given by $\langle z_1, Qz_2\rangle = \iint_{\mathcal{O}^2} q(x-y) z_1(x) z_2(y)\,\mathrm{d}x\,\mathrm{d}y$

(ii) $\dot{W}^Q(t,x) \sim \mathcal{N}(0, q(0))$ for $t \geq 0, x \in \mathcal{O}$

(iii) $\mathbb{E}[\dot{W}^Q(t,x)\dot{W}^Q(s,y)] = \delta(t-s)q(x-y)$ for $s,t \geq 0, x,y \in \mathcal{O}$

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- Intuition: a weak solution to the stochastic equation $\dot{u}(t,x) = \Delta u(t,x) + \dot{W}^Q(t,x)$ with $u(0) \equiv 0$ is given by the stochastic convolution

$$u(t,\cdot) = \int_0^t S(t-s)\dot{W}^Q(s,\cdot)\,\mathrm{d}s$$

- Problem: $\dot{W}^Q$ only exists in a distributional sense

⇝ we need to interpret the stochastic convolution as an appropriate stochastic integral

$$u(t) = \int_0^t S(t-s)\,\mathrm{d}W_s^Q$$

Hilbert space valued Wiener process

- Let Q be a nonnegative symmetric operator on $\mathcal{H}$ and $(e_k)_{k\in\mathbb{N}}$ be an ONB of $(\ker Q)^\perp$ consisting of Q-eigenvectors e_k with eigenvalues $q_k > 0$, i.e.,
  - $Qe_k = q_k e_k$ for all $k \in \mathbb{N}$
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- let $(\beta_k)_{k\in\mathbb{N}}$ be a sequence of independent Brownian motions and set

$$W^Q(t) = \sum_{k=1}^{\infty} \sqrt{q_k} e_k \beta_k(t), \quad t \in [0, T].$$

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- if $\operatorname{Tr} Q = \sum_{k=1}^\infty \langle e_k, Qe_k\rangle = \sum_{k=1}^\infty q_k < \infty$, then for $W_t^{Q,n} := \sum_{k=1}^n \sqrt{q_k} e_k \beta_k(t)$ we have for $n > m \in \mathbb{N}$

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|W^{Q,n}(t) - W^{Q,m}(t)\|^2\Big] \le \sum_{k=m+1}^{n} q_k \mathbb{E}\Big[\sup_{t\in[0,T]} \beta_k(t)^2\Big] \,\mathrm{d}t \le 4 \sum_{k=m+1}^{n} q_k \mathbb{E}[\beta_k(T)^2] = 4T \sum_{k=m+1}^{n} q_k \underset{n,m\to\infty}{\longrightarrow} 0.$$

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⇝ if (and only if) $Q^{1/2}$ is Hilbert–Schmidt[1] (or, equivalently, Q is trace class), then $(W^Q(t))_{t\ge 0}$ defines a continuous $\mathcal{H}$-valued process on $[0, T]$ and we call it a Q-Wiener process

---

[1] that is, for some (and then for any) complete orthonormal basis $(\tilde{e}_k)_{k\in\mathbb{N}}$ it holds that $\infty > \sum_{k=1}^{\infty} \|Q^{1/2}\tilde{e}_k\|^2 = \sum_{k=1}^{\infty} \langle \tilde{e}_k, Q\tilde{e}_k\rangle$, in which case the Hilbert–Schmidt norm is defined by $\|Q^{1/2}\|_{\mathrm{HS}}^2 := \sum_{k=1}^{\infty} \|Q^{1/2}\tilde{e}_k\|^2$

Q-Wiener process

- the $\mathcal{H}$-valued Q-Wiener process $W^Q(t) = \sum_{k=1}^{\infty} \sqrt{q_k} e_k \beta_k(t)$ has the following properties
  - $W^Q(0) = 0$;
  - the trajectories $t \mapsto W^Q(t)$ are $\mathbb{P}$-a.s. continuous
  - the increments $W^Q(t_2) - W^Q(t_1), \ldots, W^Q(t_n) - W^Q(t_{n-1})$ are independent for all $0 \leq t_1 < t_2 < \cdots < t_n \leq T$ and $n \in \mathbb{N}$
  - for any $0 \leq s \leq t \leq T$, $W^Q(t) - W^Q(s) \overset{d}{=} W^Q(t-s)$
  - for any $t \in [0, T]$, $W^Q(t) \sim \mathcal{N}(0, tQ)$, that is, $W^Q(t)$ is a Gaussian random variable[2] on $\mathcal{H}$ with mean 0 and covariance operator Q

[2] generally, we say that $X \sim \mathcal{N}(m, \widetilde{Q})$ for some $m \in \mathcal{H}$ and a trace-class, symmetric nonnegative operator $\widetilde{Q}$, if for any $z_1, \ldots, z_n \in \mathcal{H}$, $n \in \mathbb{N}$ it holds that

$$(\langle X, z_i \rangle)_{i=1}^n \sim \mathcal{N}\Big((\langle m, z_i \rangle)_{i=1}^n, (\langle z_j, \widetilde{Q} z_i \rangle)_{i,j=1}^n \Big)$$

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- the $\mathcal{H}$-valued Q-Wiener process $W^Q(t) = \sum_{k=1}^{\infty} \sqrt{q_k} e_k \beta_k(t)$ has the following properties
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  - the trajectories $t \mapsto W^Q(t)$ are $\mathbb{P}$-a.s. continuous
  - the increments $W^Q(t_2) - W^Q(t_1), \dots, W^Q(t_n) - W^Q(t_{n-1})$ are independent for all $0 \le t_1 < t_2 < \dots < t_n \le T$ and $n \in \mathbb{N}$
  - for any $0 \le s \le t \le T$, $W^Q(t) - W^Q(s) \stackrel{d}{=} W^Q(t-s)$
  - for any $t \in [0, T]$, $W^Q(t) \sim \mathcal{N}(0, tQ)$, that is, $W^Q(t)$ is a Gaussian random variable[2] on $\mathcal{H}$ with mean 0 and covariance operator Q
- conversely, any $\mathcal{H}$-valued process W^Q with the above properties can be decomposed as $W^Q(t) = \sum_{k=1}^{\infty} \sqrt{q_k} e_k \beta_k(t)$ by setting $\beta_k(t) = \frac{\langle W^Q(t), e_k \rangle}{\sqrt{q_k}}$

[2]generally, we say that $X \sim \mathcal{N}(m, \widetilde{Q})$ for some $m \in \mathcal{H}$ and a trace-class, symmetric nonnegative operator $\widetilde{Q}$, if for any $z_1, \dots, z_n \in \mathcal{H}$, $n \in \mathbb{N}$ it holds that

$$(\langle X, z_i \rangle)_{i=1}^n \sim \mathcal{N}\Big((\langle m, z_i \rangle)_{i=1}^n, (\langle z_j, \widetilde{Q} z_i \rangle)_{i,j=1}^n \Big)$$

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- for $Q = I$ ("integrated white noise") the series does not converge a.s. or in $L^2(\mathbb{P})$ since by the LLN

$$\|W^{I,N}(t)\|^2 = \sum_{k=1}^{N} |\beta_k(t)|^2 \asymp N\mathbb{E}[|\beta_1(t)|^2] = Nt \quad \text{as } N \to \infty$$

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Cylindrical Wiener process

- we have just seen that the cylindrical Wiener process

$$W(t) = \sum_{k\in\mathbb{N}} \beta_k(t)e_k, \quad t \in [0, T],$$

cannot be interpreted as an $\mathcal{H}$-valued random variable

[3] we write $\mathcal{L}_2(\mathcal{H}, \mathcal{H}_1)$ for the class of Hilbert–Schmidt operators from $\mathcal{H}$ to $\mathcal{H}_1$, which are defined as linear bounded operators $L\colon \mathcal{H} \to \mathcal{H}_1$ such that $\|L\|^2_{\mathcal{L}_2(\mathcal{H},\mathcal{H}_1)} = \sum_{k\in\mathbb{N}} \|Le_k\|^2_{\mathcal{H}_1} < \infty$ for some (any) orthonormal basis (e_k) of $\mathcal{H}$

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- however, if for another Hilbert space $\mathcal{H}_1$, $L\colon\mathcal{H}\to\mathcal{H}_1$ is Hilbert–Schmidt[3], then
$$LW(t):=\sum_{k\in\mathbb{N}}\beta_k(t)Le_k,\quad t\in[0,T]$$
defines an a.s. continuous $\mathcal{H}_1$-valued process since for $S_n=\sum_{k=1}^n\beta_k Le_k$ and $n>m$,

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}\|S_n(t)-S_m(t)\|^2_{\mathcal{H}_1}\Big]\le 4\sup_{t\in[0,T]}\mathbb{E}\big[\|S_n(t)-S_m(t)\|^2_{\mathcal{H}_1}\big] &= 4\sup_{t\in[0,T]}\sum_{k=m+1}^n\sum_{l=m+1}^n\langle Le_k,Le_l\rangle_{\mathcal{H}_1}\underbrace{\mathbb{E}[\beta_k(t)\beta_l(t)]}_{=\mathbb{E}[\beta_1(t)^2]\mathbf{1}_{\{k=l\}}}\\ &= 4\sup_{t\in[0,T]}\mathbb{E}[\beta_1(t)^2]\sum_{k=m+1}^n\|Le_k\|^2_{\mathcal{H}_1}\\ &= 4T\sum_{k=m+1}^n\|Le_k\|^2_{\mathcal{H}_1}\underset{n,m\to\infty}{\longrightarrow}0.\end{aligned}$$

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$$\begin{aligned} C((t, z), (t', z')) &:= \operatorname{Cov}\left(\langle W(t), z\rangle, \langle W(t'), z'\rangle\right) \\ &= \sum_{k,l\in\mathbb{N}} \langle z, e_k\rangle\langle z', e_l\rangle \mathbb{E}[\beta_k(t)\beta_l(t')] \\ &= (t \wedge t') \sum_{k\in\mathbb{N}} \langle z, e_k\rangle\langle z', e_k\rangle \\ &= (t \wedge t')\langle z, z'\rangle. \end{aligned}$$

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⇝ if z, z' satisfy $\|z\| = \|z'\| = 1$ and are orthogonal, then $(\langle W(t), z\rangle)_{t\in[0,T]}$ and $(\langle W(t), z'\rangle)_{t\in[0,T]}$ are independent Brownian motions.

The stochastic integral

- let Q be trace class (i.e., $\sum_k q_k < \infty$) or $Q = I$ (i.e., $q_k = 1$) and consider a Q-, resp. cylindrical $\mathcal{H}$-valued Wiener process $W^Q = \sum_{k\in\mathbb{N}} \beta_k e_k^0$ for $e_k^0 = \sqrt{q_k} e_k$, which is an ONB of $\mathcal{H}_0 = Q^{1/2}(\mathcal{H})$ endowed with $\langle u, v\rangle_0 = \langle Q^{-1}u, v\rangle$
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Moreover for any $z \in \mathcal{U}$ we obtain the linearity property

$$\Big\langle \int_0^t \Phi(s)\,\mathrm{d}W^Q(s), z\Big\rangle_{\mathcal{U}} = \sum_{k\in\mathbb{N}} \int_0^t \sum_{i\in I} \langle \Phi(s)e_k^0, \tilde{e}_i\rangle_{\mathcal{U}} \langle \tilde{e}_i, z\rangle_{\mathcal{U}} \,\mathrm{d}\beta_k(s)$$

$$= \sum_{k\in\mathbb{N}} \int_0^t \langle \Phi(s)e_k^0, z\rangle_{\mathcal{U}} \,\mathrm{d}\beta_k(s)$$

$$= \int_0^t \langle \Phi(s)\cdot, z\rangle_{\mathcal{U}} \,\mathrm{d}W^Q(s).$$

For $\mathcal{U} = \mathcal{H}$, continuing the calculation yields

$$\Big\langle \int_0^t \Phi(s)\,\mathrm{d}W^Q(s), z\Big\rangle = \int_0^t \langle \Phi^*(s)z, \cdot\rangle \,\mathrm{d}W^Q(s) =: \int_0^t \langle \Phi^*(s)z, \mathrm{d}W^Q(s)\rangle$$

Linear stochastic evolution equations with additive noise

Weak solutions of linear equations with additive noise

Let A be the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on $\mathcal{H}$ and W^Q be an $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$-adapted process, which is either a Q-Wiener process or, for $Q = I$, the cylindrical Wiener process. A weak solution to the stochastic evolution equation

$$\begin{cases} \mathrm{d}X(t) = AX(t)\,\mathrm{d}t + \mathrm{d}W^Q(t) \\ X(0) = \xi \in \mathcal{F}_0, \end{cases}$$

is an $\mathbb{F}$-predictable process with a.s. integrable paths such that a.s.

$$\forall z \in D(A^*), t \in [0,T] \;:$$

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Theorem

Suppose that $\int_0^T \|S(t)\|^2_{\mathcal{L}_2(\mathcal{H}_0,\mathcal{H})}\,\mathrm{d}t < \infty$. Then, the unique weak solution is given by the mild solution

$$X(t) = S(t)\xi + \int_0^t S(t-s)\,\mathrm{d}W^Q(s), \quad t \in [0,T],$$

where $W_A^Q(\cdot) := \int_0^{\cdot} S(\cdot - s)\,\mathrm{d}W^Q(s)$ is called the stochastic convolution.

Proof (Sketch). Assume for simplicity $\xi = 0$. Then,

$$\begin{aligned}
&\int_0^t \langle X(s), A^* z\rangle\,\mathrm{d}s = \int_0^t \langle A^* z, W_A^Q(s)\rangle\,\mathrm{d}s \\
&= \int_0^t \int_0^s \langle S^*(s-r)A^* z, \mathrm{d}W^Q(r)\rangle\,\mathrm{d}s = \int_0^t \Big\langle \int_r^t S^*(s-r)A^* z\,\mathrm{d}s, \mathrm{d}W^Q(r)\Big\rangle \\
&= \int_0^t \Big\langle \int_0^{t-r} \frac{\mathrm{d}}{\mathrm{d}s} S^*(s) z\,\mathrm{d}s, \mathrm{d}W^Q(r)\Big\rangle = \int_0^t \langle S^*(t-r)z, \mathrm{d}W(r)\rangle - \langle W^Q(t), z\rangle \\
&= \langle W_A^Q(t), z\rangle - \langle W^Q(t), z\rangle = \langle X(t), z\rangle - \langle W^Q(t), z\rangle.
\end{aligned}$$

□

The stochastic heat equation

- Suppose that $\mathcal{O}$ is bounded. We can now finally discuss weak solutions of the stochastic heat equation

$$\mathrm{d}X(t) = \Delta X(t) + \mathrm{d}W^Q(t)$$

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- suppose $\operatorname{Tr} Q < \infty$. Since $(S(t))_{t\geq 0}$ is a self-adjoint semigroup with $\|S(t)\| \leq 1$, we have

$$\|S(t)Q^{1/2}\|^2_{\mathrm{HS}} = \operatorname{Tr} Q^{1/2}S(t)^2Q^{1/2} = \operatorname{Tr} S(2t)Q \leq \|S(2t)\| \operatorname{Tr} Q \leq \operatorname{Tr} Q,$$

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which is integrable, whence the mild solution exists and is the weak solution.
- if $Q = I$, let $(e'_k)_{k\in\mathbb{N}}$ an orthonormal eigenbasis of $-\Delta$ with eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ having Weyl asymptotics $\lambda_k \asymp k^{2/d}$. Then
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- for $Q = I$ and deterministic $X(0)$, the tested solutions $(\langle X(\cdot), e'_k\rangle)_{k\in\mathbb{N}}$ are independent Ornstein–Uhlenbeck processes: for $\beta'_k(s) := \langle W(s), e'_k\rangle$,

$$\langle X(t), e'_k\rangle = \langle X(0), S(t)e'_k\rangle + \int_0^t \left\langle S(t-s)e'_k, \mathrm{d}W(s)\right\rangle = \mathrm{e}^{-\lambda_k t}\langle X(0), e'_k\rangle + \int_0^t \mathrm{e}^{-\lambda_k(t-s)}\,\mathrm{d}\beta'_k(s).$$