Change point estimation for a stochastic heat equation

SPDES: Statistics meets Numerics – Institut Mittag-Leffler

Lukas Trottner

based on joint works with Markus Reiß, Claudia Strauch and Anton Tiepner

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Humboldt University of Berlin   Heidelberg University   Free University of Berlin

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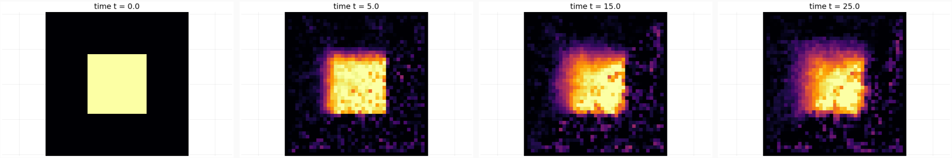
Change point model for stochastic heat equations



- Stochastic heat equation

$$\mathrm{d}X(t) = \Delta_\vartheta X(t)\,\mathrm{d}t + \mathrm{d}W(t), \quad \Delta_\vartheta = \nabla\cdot\vartheta\nabla,$$

with Dirichlet boundary conditions, and broken diffusivity

$$\vartheta(x) = \vartheta_- \mathbf{1}_{\Lambda_-}(x) + \vartheta_+ \mathbf{1}_{\Lambda_+}(x), \quad x \in [0,1]^d = \Lambda_- \uplus \Lambda_+, \vartheta_- \wedge \vartheta_+ > 0.$$

- special case for $d = 1$: $\Lambda_+ = (\tau, 1]$ with change point τ

The univariate case

$-\Delta_\vartheta$ is induced by Dirichlet form

$$\mathcal{E}(u,v) := \langle \vartheta \partial_x u, \partial_x v \rangle = \int_0^1 \vartheta(x)\, \partial_x u(x)\, \partial_x v(x)\, \mathrm{d}x, \qquad u, v \in H_0^1((0,1)),$$

and generates C_0-semigroup $S_\vartheta(t) = \exp(t\Delta_\vartheta)$, $t \in [0,T]$, having transition densities that satisfy the heat kernel bound

$$p_t^\vartheta(x,y) \le c_1 t^{-1/2} \exp\Big(-\frac{|x-y|^2}{c_2 t}\Big), \quad (x,y) \in (0,1)^2, t \in (0,T].$$

Mild solution

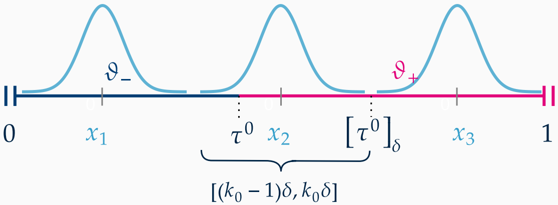
$$X(t) = \int_0^t S_\vartheta(t-s)\, \mathrm{d}W(s), \quad t \in [0,T], \qquad \text{(assume } X(0) \equiv 0\text{)}$$

is $L^2((0,1))$-valued and we have

$$\langle X(t), z \rangle = \int_0^t \langle X(s), \Delta_\vartheta z \rangle\, \mathrm{d}s + \langle W(t), z \rangle, \quad \forall\, z \in D(\Delta_\vartheta) = \big\{u \in H_0^1((0,1)) : \vartheta \partial_x u \in H^1((0,1))\big\}.$$

Observation model

- let $K : \mathbb{R} \to \mathbb{R}$ be a smooth kernel with supp $K \subset [-1/2, 1/2]$, $\|K\|_{L^2} = 1$ and for $\delta = n^{-1}$, $x_i = (i - 1/2)\delta\,(i \in \{1, \dots, \delta^{-1}\})$, define $K_{\delta,i}(x) = \delta^{-1/2} K(\delta^{-1}(x - x_i))$
- local observations $(X_{\delta,i}(t))_{t\in[0,T]} = (\langle X(t), K_{\delta,i}\rangle)_{t\in[0,T]}$ and $(X^{\Delta}_{\delta,i}(t))_{t\in[0,T]} = (\langle X(t), \Delta K_{\delta,i}\rangle)_{t\in[0,T]}$



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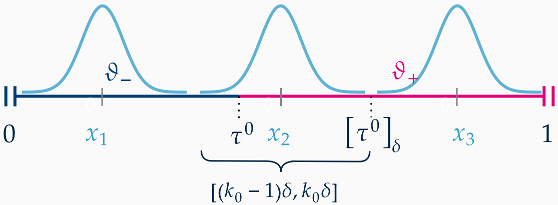

- we have

$$X_{\delta,i}(t) = \begin{cases} \int_0^t \vartheta^0_{\pm} X^{\Delta}_{\delta,i}(s)\,\mathrm{d}s + B_{\delta,i}(t), & i \gtrless k_0, \\ \int_0^t \int_0^s \langle \Delta_{\vartheta^0} S_{\vartheta^0}(s-u) K_{\delta,i}, \mathrm{d}W(u)\rangle\,\mathrm{d}s + B_{\delta,k_0}(t), & i = k_0, \end{cases}$$

for independent Brownian motions $(B_{\delta,i})_{i\in[\delta^{-1}]}$

Estimation approach

- modified local log-likelihood:

$$\ell_{\delta,i}(\vartheta_-,\vartheta_+,\vartheta_\circ,k) := \vartheta_{\delta,i}(k)\int_0^T X^\Delta_{\delta,i}(t)\,\mathrm{d}X_{\delta,i}(t) - \frac{\vartheta_{\delta,i}(k)^2}{2}\int_0^T X^\Delta_{\delta,i}(t)^2\,\mathrm{d}t, \quad \vartheta_{\delta,i}(k) := \begin{cases}\vartheta_-, & i<k,\\ \vartheta_\circ, & i=k,\\ \vartheta_+, & i>k\end{cases}$$

- set $(\hat\vartheta_-,\hat\vartheta_+,\hat\vartheta_\circ,\hat\tau) := (\hat\vartheta_-,\hat\vartheta_+,\hat\vartheta_\circ,\hat k\delta)$, where

$$(\hat\vartheta_-,\hat\vartheta_+,\hat\vartheta_\circ,\hat k) := \underset{(\vartheta_-,\vartheta_+,\vartheta_\circ,k)}{\arg\max}\sum_{i\in[\delta^{-1}]}\ell_{\delta,i}(\vartheta_-,\vartheta_+,\vartheta_\circ,k)$$

$$= \underset{(\vartheta_-,\vartheta_+,\vartheta_\circ,k)}{\arg\min}\Big\{\frac{1}{2}\sum_{i=1}^{\delta^{-1}}(\vartheta_{\delta,i}(k)-\vartheta^0_{\delta,i})^2 I_{\delta,i} - \sum_{i=1}^{\delta^{-1}}(\vartheta_{\delta,i}(k)-\vartheta^0_{\delta,i})M_{\delta,i} - \vartheta_{\delta,k_0}(k)R_{\delta,k_0}(\vartheta^0_\circ)\Big\},$$

for

$$M_{\delta,i} := \int_0^T X^\Delta_{\delta,i}(t)\,\mathrm{d}B_{\delta,i}(t), \quad I_{\delta,i} := \int_0^T X^\Delta_{\delta,i}(t)^2\,\mathrm{d}t,$$

and $R_{\delta,k_0}(\vartheta^0_\circ)$ is an error term resulting from $K_{\delta,k_0}\notin D(\Delta_\vartheta)$ in general

Basic estimates

Lemma (Reiß, Strauch and T., 2023+)

- For any $i \in [\delta^{-1}] \setminus \{k_0\}$,

$$\mathbb{E}[I_{\delta,i}] = \frac{T}{2\vartheta(x_i)} \|K'\|_{L^2}^2 \delta^{-2} + \mathcal{O}(1),$$

  and, moreover, $\mathbb{E}[I_{\delta,k_0}] \sim \delta^{-2}$;

- for any vector $\alpha \in \mathbb{R}^n$ s.t. $\alpha_{k_0} = 0$,

$$\mathrm{Var}\Big(\sum_{i=1}^{\delta^{-1}} \alpha_i I_{\delta,i} \Big) \leq \frac{T}{2\underline{\vartheta}^3} \delta^{-2} \|\alpha\|_{\ell^2}^2 \|K'\|_{L^2}^2;$$

-

$$\mathbb{E}[|R_{\delta,k_0}(\vartheta_\circ)|] \lesssim \delta^{-2}, \quad \mathrm{Var}(R_{\delta,k_0}(\vartheta_\circ)) \lesssim \delta^{-2},$$

  and, moreover,

$$\exists\, \vartheta_\circ^0 : \quad |\mathbb{E}[R_{\delta,k_0}(\vartheta_\circ^0)]| \leq \delta^{-1}.$$

Concentration results

$\sum_{i=1}^{\delta^{-1}} \alpha_i (I_{\delta,i} - \mathbb{E}[I_{\delta,i}])$ belongs to second Wiener chaos for an isonormal Gaussian process associated to $(X_i^\Delta(t))_{t\in[0,T],i\in[\delta^{-1}]}$ $\rightsquigarrow$ relate to Bernstein-type concentration result from Nourdin and Viens (2009)[1]

Proposition (Reiß, Strauch and T., 2023+)

Let $\alpha \in \mathbb{R}_+^n \setminus \{0\}$ s.t. $\alpha_{k_0} = 0$. Then, for any $z > 0$, we have

$$\mathbb{P}\Big(\Big|\sum_{i=1}^n \alpha_i (I_{\delta,i} - \mathbb{E}[I_{\delta,i}])\Big| \geq z\Big) \leq 2\exp\Big(-\frac{\underline{\vartheta}^2}{4\|\alpha\|_\infty}\frac{z^2}{z + \sum_{i=1}^n \alpha_i \mathbb{E}[I_{\delta,i}]}\Big).$$

[1] Nourdin, I., and F.G. Viens (2009). Density formula and concentration inequalities with Malliavin calculus. *Electron. J. Prob.*

Concentration results

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Proposition (Reiß, Strauch and T., 2023+)

Let

$$\overline{M}_{\delta,i} := \int_0^{\sigma_i} X_{\delta,i}^{\Delta}(t)\,\mathrm{d}B_{\delta,i}(t), \quad \text{where } \sigma_i := \inf\Big\{t \geq 0 \,:\, \int_0^t X_{\delta,i}(s)^2\,\mathrm{d}s > \mathbb{E}[I_{\delta,i}]\Big\}.$$

Then, $(\overline{M}_{\delta,i})_{i\in[\delta^{-1}]} \sim N\big(0, \mathrm{diag}((\mathbb{E}[I_{\delta,i}])_{i\in[\delta^{-1}]})\big)$ and

$$\mathbb{P}\Big(\Big|\sum_{i=1}^n \alpha_i (M_{\delta,i} - \overline{M}_{\delta,i})\Big| \geq z, \sum_{i=1}^n \alpha_i^2 |I_{\delta,i} - \mathbb{E}[I_{\delta,i}]| \leq L\Big) \leq \exp(-z^2/2L), \quad \alpha \in \mathbb{R}^n, z, L > 0$$

Rate of convergence

Define the jump height $\eta := \vartheta_+^0 - \vartheta_-^0$.

Theorem (Reiß, Strauch and T., 2023+)

Suppose that $\vartheta_\pm^0 \xrightarrow[\delta \to 0]{} \vartheta_\pm^*$ and that $|\eta| \geq \underline{\eta} > 0$ for all $\delta \in 1/\mathbb{N}$. Then,

$$|\hat{\tau} - \tau^0| = \mathcal{O}_{\mathbb{P}}(\delta) \quad \text{and} \quad |\hat{\vartheta}_\pm - \vartheta_\pm^0| = \mathcal{O}_{\mathbb{P}}(\delta^{3/2}).$$

- the estimation rate for τ^0 cannot be improved due to discretisation effects
- the estimation rate for $\vartheta_\pm^0$ is the minimax optimal rate for parametric estimation from multiple local measurements in the model $A_\vartheta = \vartheta\Delta$ without change point[1]

[1] Altmeyer, R., Tiepner, A. and M. Wahl (2024). Optimal parameter estimation for linear SPDEs from multiple measurements. *Ann. Stat.*

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Suppose that $\vartheta_\pm^0 \underset{\delta\to 0}{\longrightarrow} \vartheta_\pm^*$ and that $|\eta| \geq \underline{\eta} > 0$ for all $\delta \in 1/\mathbb{N}$. Then,

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Proof outline:

1. verify basic consistency of $(\hat{\vartheta}_\pm, \hat{\tau})$
2. determine appropriate empirical process $(\mathcal{L}_\delta)_{\delta\in 1/\mathbb{N}}$ with $[\underline{\vartheta}, \overline{\vartheta}]^3 \times (0,1] \ni \chi \mapsto \mathcal{L}_\delta(\chi)$ such that
$$(\hat{\vartheta}_-, \hat{\vartheta}_+, \hat{\vartheta}_\circ, \hat{\tau}) \in \underset{\chi\in[\underline{\vartheta},\overline{\vartheta}]^3\times(0,1]}{\arg\min} \mathcal{L}_\delta(\chi)$$
3. control local fluctuations of centered empirical process $\mathcal{L}_\delta - \widetilde{\mathcal{L}}_\delta(\chi)$ around χ^0, where $\widetilde{\mathcal{L}}_\delta(\chi) = \mathbb{E}[\mathcal{L}_\delta(\chi)] + \mathcal{O}(\delta^2)$
4. exploit (non-standard) peeling device to prove convergence rate

Vanishing signal

- for the previous consistency result it was crucial that the jump height η does not vanish
- assume now that $\eta \underset{\delta\to 0}{\longrightarrow} 0$ and that $\vartheta^0_\pm = \vartheta^0_\pm(\delta)$ are known
- set $\hat{\tau} = \hat{k}\delta$, where

$$\begin{aligned}\hat{k} := \underset{k=1,\dots,\delta^{-1}}{\arg\max} \sum_{i=1}^{k} \Big(\vartheta^0_- \int_0^T X^\Delta_{\delta,i}(t)\,\mathrm{d}X_{\delta,i}(t) - \frac{(\vartheta^0_-)^2}{2} \int_0^T X^\Delta_{\delta,i}(t)^2\,\mathrm{d}t\Big) \\ + \sum_{i=k+1}^{\delta^{-1}} \Big(\vartheta^0_+ \int_0^T X^\Delta_{\delta,i}(t)\,\mathrm{d}X_{\delta,i}(t) - \frac{(\vartheta^0_+)^2}{2} \int_0^T X^\Delta_{\delta,i}(t)^2\,\mathrm{d}t\Big) \\ = \underset{k=1,\dots,\delta^{-1}}{\arg\min}\, Z_k,\end{aligned}$$

for

$$Z_k = \begin{cases} 0, & k = k_0, \\ -\eta \sum_{i=k+1}^{k_0} \int_0^T X^\Delta_{\delta,i}(t)\,\mathrm{d}B_{\delta,i}(t) + \frac{\eta^2}{2} \sum_{i=k+1}^{k_0} \int_0^T X^\Delta_{\delta,i}(t)^2\,\mathrm{d}t + \eta R_{\delta,k_0}(\vartheta^0_-), & k < k_0, \\ \eta \sum_{i=k_0+1}^{k} \int_0^T X^\Delta_{\delta,i}(t)\,\mathrm{d}B_{\delta,i}(t) + \frac{\eta^2}{2} \sum_{i=k_0+1}^{k} \int_0^T X^\Delta_{\delta,i}(t)^2\,\mathrm{d}t, & k > k_0, \end{cases}.$$

Limit theorem for vanishing signal

Reformulate the estimator again in terms of an M-estimator: Let $v_\delta \to 0$, and define

$$M_{T,\delta}^{\tau^0}(h) = M_{T,\delta}(\tau^0 + hv_\delta) - M_{T,\delta}(\tau^0), \quad \text{for } M_{T,\delta}(z) := \sum_{i=1}^{\lceil z/\delta \rceil} M_{\delta,i},\ z \in [0,1]$$

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s.t.

$$\mathcal{Z}_\delta(v_\delta^{-1}(\hat{\tau} - \tau^0)) = \min_{h \in [-\tau_0/v_\delta, (1-\tau^0)/v_\delta]} \mathcal{Z}_\delta(h) + \mathcal{O}_{\mathbb{P}}(\eta^2\delta^{-2}), \quad \text{for } \mathcal{Z}_\delta(h) := \eta M_{T,\delta}^{\tau^0}(h) + \frac{\eta^2}{2} I_{T,\delta}^{\tau^0}(h)$$

Limit theorem for vanishing signal

Reformulate the estimator again in terms of an M-estimator: Let $v_\delta \to 0$, and define

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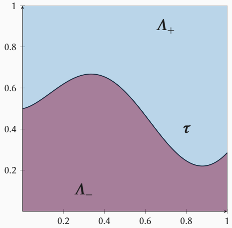
Theorem (Reiß, Strauch and T., 2023+)

Assume $\eta = o(\delta)$ and $\delta^{3/2} = o(\eta)$. Then, for a two-sided Brownian motion $(B^{\leftrightarrow}(h),\ h \in \mathbb{R})$, we have

$$\underbrace{\frac{\eta^2}{\delta^3}}_{=v_\delta^{-1}} \frac{T\|K'\|^2_{L^2}}{2\vartheta^*}(\hat{\tau} - \tau^0) \xrightarrow{\mathrm{d}} \operatorname*{arg\,min}_{h \in \mathbb{R}} \left\{ B^{\leftrightarrow}(h) + \frac{|h|}{2} \right\}, \quad \text{as } \delta \to 0.$$

The multivariate case



- Recall:

$$\mathrm{d}X(t) = \Delta_\vartheta X(t)\,\mathrm{d}t + \mathrm{d}W(t), \quad \Delta_\vartheta = \nabla \cdot \vartheta \nabla,$$

  with

$$\vartheta(x) = \vartheta_- \mathbf{1}_{\Lambda_-}(x) + \vartheta_+ \mathbf{1}_{\Lambda_+}(x), \quad x \in [0,1]^d = \Lambda_- \uplus \Lambda_+, \vartheta_- \wedge \vartheta_+ > 0.$$

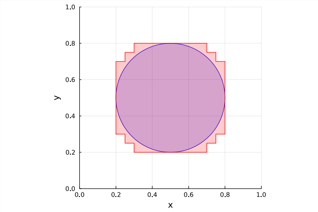
- we call Λ_+ a change domain
- structural similarities to image reconstruction problem

$$Y_i = \vartheta_- \mathbf{1}_{\Lambda_-}(X_i) + \vartheta_+ \mathbf{1}_{\Lambda_+}(X_i) + \varepsilon_i,$$

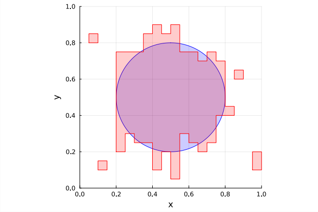
  for (possibly random) measurement locations X_i and noise ε_i

Local observations

- put regular δ-grid on $[0,1]^d$ with grid centers x_α, $\alpha \in [n]^d = [\delta^{-1}]^d$ and aim for estimation of minimal tiling $\Lambda_+^{\updownarrow}$ of Λ_+^0



$\Lambda_+^{\updownarrow}$



$\widehat{\Lambda}_+$

- set $K_{\delta,\alpha} = \delta^{-d/2} K((\cdot - x_\alpha)/\delta)$
- local observations $X_{\delta,\alpha}(t) = \langle X(t), K_{\delta,\alpha}\rangle$ and $X^{\Delta}_{\delta,\alpha}(t) = \langle X(t), \Delta K_{\delta,\alpha}\rangle$ given for $\alpha \in [n]^d$, $t \in [0,T]$

Estimation approach

- $\mathcal{A}_+$ is a family of tiling sets such that $\Lambda_+^{\updownarrow} \in \mathcal{A}_+$
- $\Theta_\pm$ are $\underline{\eta}$-separated sets such that $\vartheta_\pm^0 \in \Theta_\pm$
- set

$$(\hat{\vartheta}_-, \hat{\vartheta}_+, \widehat{\Lambda}_+) \in \underset{(\vartheta_-,\vartheta_+,\Lambda_+)\in\Theta_-\times\Theta_+\times\mathcal{A}_+}{\arg\max} \sum_{\alpha\in[n]^d} \ell_{\delta,\alpha}(\vartheta_-, \vartheta_+, \Lambda_+),$$

where

$$\ell_{\delta,\alpha}(\vartheta_-, \vartheta_+, \Lambda_+) = \vartheta_{\delta,\alpha}(\Lambda_+) \int_0^T X_{\delta,\alpha}^{\Delta}(t)\, \mathrm{d}X_{\delta,\alpha}(t) - \frac{\vartheta_{\delta,\alpha}(\Lambda_+)^2}{2} \int_0^T X_{\delta,\alpha}^{\Delta}(t)^2\, \mathrm{d}t,$$

for

$$\vartheta_{\delta,\alpha}(\Lambda_+) = \begin{cases} \vartheta_+, & x_\alpha \in \Lambda_+, \\ \vartheta_-, & \text{else.} \end{cases}$$

Convergence rate

Theorem (Tiepner and T., 2025+)

Suppose that the number of hypercubes intersecting $\partial\Lambda_+^0$ is of order $\delta^{-d+\beta}$ for some $\beta \in (0,1]$. Then,

$$\mathbb{E}\big[\operatorname{vol}_d(\widehat{\Lambda}_+ \triangle \Lambda_+^0)\big] \lesssim \delta^{\beta},$$

and $\hat{\vartheta}_\pm$ are consistent.

In particular, if

- Λ_+^0 epigraph of a β-Hölder function $\implies \mathbb{E}\big[\operatorname{vol}_d(\widehat{\Lambda}_+ \triangle \Lambda_+^0)\big] = \mathbb{E}\big[\|\hat{\tau} - \tau^0\|_{L^1}\big] \lesssim \delta^{\beta}$;
- Λ_+^0 convex $\implies \mathbb{E}\big[\operatorname{vol}_d(\widehat{\Lambda}_+ \triangle \Lambda_+^0)\big] \lesssim \delta$

and we can choose $\mathcal{A}_+$ s.t. in the first case $|\mathcal{A}_+| \asymp \delta^{-d}$ and in the second case $|\mathcal{A}_+| \asymp \delta^{-(d+1)}$.

Improved convergence results when there is no geometric bias

- assume now that Λ_+^0 can be perfectly built from hypercubes, i.e., $\overline{\Lambda_+^0} \in \mathcal{A}_+$
- for fixed $\Lambda_+ \in \mathcal{A}_+$, the function $\mathbb{R}^2 \ni (\vartheta_-, \vartheta_+) \mapsto \sum_{\alpha \in [n]^d} \ell_{\delta,\alpha}(\vartheta_-, \vartheta_+, \Lambda_+)$ is maximised in

$$\vartheta_\pm^{\Lambda_+} = \frac{\sum_{\mathrm{Sq}(\alpha)^\circ \subset \Lambda_\pm} \int_0^T X_{\delta,\alpha}^\Delta(t)\,\mathrm{d}X_{\delta,\alpha}(t)}{\sum_{\mathrm{Sq}(\alpha)^\circ \subset \Lambda_\pm} I_{\delta,\alpha}}$$

⇝ set

$$\widetilde{\Lambda}_+ \in \underset{\Lambda_+ \in \mathcal{A}_+}{\arg\max} \sum_{\alpha \in [n]^d} \ell_{\delta,\alpha}(\vartheta_\pm^{\Lambda_+}, \Lambda_+), \quad \widetilde{\Lambda}_- = \widetilde{\Lambda}_+^c, \quad \tilde{\vartheta}_\pm = \vartheta_\pm^{\widetilde{\Lambda}_+}$$

and (for identification purposes)

$$\widehat{\Lambda}_+^* = \begin{cases} \widetilde{\Lambda}_+, & \text{if } \operatorname{vol}(\widetilde{\Lambda}_+ \cap \widehat{\Lambda}_+) \geq \operatorname{vol}(\widetilde{\Lambda}_+ \cap \widehat{\Lambda}_-) \\ \operatorname{cl} \widetilde{\Lambda}_-, & \text{else} \end{cases}, \quad \hat{\vartheta}_\pm^* = \begin{cases} \tilde{\vartheta}_\pm, & \text{if } \widehat{\Lambda}_+^* = \widetilde{\Lambda}_\pm, \\ \tilde{\vartheta}_\mp, & \text{else} \end{cases}$$

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⇝ set

$$\widetilde{\Lambda}_+ \in \arg\max_{\Lambda_+ \in \mathcal{A}_+} \sum_{\alpha \in [n]^d} \ell_{\delta,\alpha}(\vartheta_\pm^{\Lambda_+}, \Lambda_+), \quad \widetilde{\Lambda}_- = \widetilde{\Lambda}_+^c, \quad \tilde{\vartheta}_\pm = \vartheta_\pm^{\widetilde{\Lambda}_+}, \quad \text{for } \vartheta_\pm^{\Lambda_+} = \frac{\sum_{\mathrm{Sq}(\alpha)^\circ \subset \Lambda_\pm} \int_0^T X_{\delta,\alpha}^\Delta(t)\,\mathrm{d}X_{\delta,\alpha}(t)}{\sum_{\mathrm{Sq}(\alpha)^\circ \subset \Lambda_\pm} I_{\delta,\alpha}}$$

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Theorem (Tiepner and T., 2025+)

Suppose that $\overline{\Lambda_+^0} \in \mathcal{A}_+$, $|\mathcal{A}_+| \lesssim \delta^{-c}$ for some $c > 0$ and $\lim_{\delta\to 0} \operatorname{vol}(\Lambda_\pm^0) = v_\pm^0 > 0$. Then,

$$\lim_{\delta \to 0} \mathbb{P}\Big(\widehat{\Lambda}_+^* = \Lambda_+^0\Big) = 1$$

and

$$\delta^{-(d/2+1)}(\hat{\vartheta}_\pm^* - \vartheta_\pm^0) \xrightarrow{d} \mathcal{N}\Big(0, \frac{2\vartheta_\pm^0}{T\|\nabla K\|_{L^2}^2 v_\pm^0}\Big).$$

If $\mathrm{cl}\,\Lambda_-^0 \notin \mathcal{A}_+$ the above continues to hold with $\widehat{\Lambda}_+^*$ and $\hat{\vartheta}_\pm$ replaced by $\widetilde{\Lambda}_+$ and $\tilde{\vartheta}_\pm$, resp.

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Thank you for your attention!