Markov additive friendships

43rd Conference on Stochastic Processes and their Applications – Lisbon 2023

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24 July 2023

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Theory of friends for Lévy processes

Wiener–Hopf factorisation of Lévy processes

- let ξ be a Lévy process with characteristic exponent ψ, i.e., $\mathbb{E}[e^{i\theta\xi_t}] = e^{t\psi(\theta)}$
- the ascending ladder height process H^+ is a subordinator tracking the new suprema of ξ
- the descending ladder height process H^- plays the same role for the infima
- for $\psi^\pm$ denoting the characteristic exponents of $H^\pm$, it holds

$$\psi(\theta) = -\psi^-(-\theta)\psi^+(\theta)$$

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Problem

Given a LK exponent ψ the Wiener–Hopf factors can very rarely be explicitly determined.

The inverse problem

- start with two subordinators $H^{\pm}$ having LK exponents $\psi^{\pm}$
- when is there a Lévy process ξ with LK exponent ψ such that $\psi = -\psi^{-}(-\cdot)\psi^{+}$?
- when such ξ exists, we call $H^{\pm}$ friends and ξ the bonding process

The Theorem of friends

- $d^{\pm}$ drift of $H^{\pm}$
- $\Pi^{\pm}$ Lévy measure of $H^{\pm}$
- $H^{\pm}$ are compatible if $d^{\mp} > 0$ implies that $\Pi^{\pm}$ has a càdlàg density $\partial\Pi^{\pm}$ that can be expressed as the tail of a signed measure

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Theorem (Vigon, 2002)

$H^{\pm}$ are friends if, and only if, they are compatible and the function

$$\Upsilon(x) = \begin{cases} \int_{(0,\infty)} (\Pi^{-}(y,\infty) - \psi^{-}(0))\,\Pi^{+}(x + dy) + d^{-}\partial\Pi^{+}(x), & x > 0, \\ \int_{(0,\infty)} (\Pi^{+}(y,\infty) - \psi^{+}(0))\,\Pi^{-}(-x + dy) + d^{+}\partial\Pi^{-}(-x), & x < 0, \end{cases}$$

is a.e. increasing on $(0,\infty)$ and a.e. decreasing on $(-\infty,0)$.

Then, Υ is a.e. the right/left tail of the Lévy measure of the bonding process.

Philanthropy

- a subordinator H^+ is called philanthropist if its Lévy measure admits a decreasing density
- equivalently, philanthropists are subordinators that are friends with pure drift subordinators $H_t^- = d^- t$

$\rightsquigarrow$ spectrally negative Lévy processes that do not drift to $-\infty$ can be factorised into a philanthropist and a pure drift

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Theorem (Vigon, 2002)

Any two philanthropists are friends.

Markov additive friendships

Markov additive processes

- a nice Markov process (ξ, J) is a MAP on $\mathbb{R} \times \{1, \dots, n\}$ if

$$\mathbb{E}^{0,i}\left[f(\xi_{t+s} - \xi_t, J_{t+s}) \mid \mathcal{F}_t\right]\mathbf{1}_{\{t<\zeta\}} = \mathbb{E}^{0,J_t}[f(\xi_s, J_s)]\mathbf{1}_{\{t<\zeta\}}$$

- equivalently, a MAP can be characterised as a regime-switching Lévy process
  - $\xi^{(i)}$ is a Lévy process for any phase $i \in [n]$
  - J is a Markov chain with transition matrix Q
  - when J is in state i, run an independent copy of $\xi^{(i)}$
  - phase switches from i to j trigger an additional jump with distribution $F_{i,j}$

Analytic characterisation of MAPs

- ψ_i LK exponent of $\xi^{(i)}$
- Π_i Lévy measure of $\xi^{(i)}$ and denote $\Pi_{i,j} \coloneqq q_{i,j} F_{i,j}$
- call $\boldsymbol{\Pi} = (\Pi_{i,j})_{i,j\in[n]}$ Lévy measure matrix of ξ
- we have $\mathbb{E}^{0,i}[e^{i\theta\xi_t}\mathbf{1}_{\{J_t=j\}}] = (\exp(t\boldsymbol{\Psi}(\theta)))_{i,j}$ with characteristic matrix exponent

$$\boldsymbol{\Psi}(\theta) = \operatorname{diag}\big((\psi_i(\theta))_{i\in[n]}\big) + \boldsymbol{Q} \odot \big(\widehat{F_{i,j}}(\theta)\big)_{i,j\in[n]}$$
$$= \begin{bmatrix} \psi_1(\theta) + q_{1,1} & \widehat{\Pi}_{1,2}(\theta) & \cdots & \widehat{\Pi}_{1,n}(\theta) \\ \widehat{\Pi}_{2,1}(\theta) & \psi_2(\theta) + q_{2,2} & \cdots & \widehat{\Pi}_{2,n}(\theta) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\Pi}_{n,1}(\theta) & \widehat{\Pi}_{n,2}(\theta) & \cdots & \psi_n(\theta) + q_{n,n} \end{bmatrix}$$

MAP Wiener–Hopf factorisation

- ascending/descending ladder height MAPs $(H^{\pm}, J^{\pm})$: ordinator $H^{\pm}$ tracks new suprema/infima of ξ and $J^{\pm}$ tracks phases during which they occur
- they are MAP subordinators (increasing ordinators) with matrix exponents $\mathbf{\Psi}^{\pm}$
- we always assume that J is irreducible and hence has an invariant distribution represented by a vector $\boldsymbol{\pi}$

Theorem [DDK17, I17, D**T**W23+]

$$\mathbf{\Psi}(\theta) = -\mathbf{\Delta}_{\boldsymbol{\pi}}^{-1}\mathbf{\Psi}^{-}(-\theta)\mathbf{\Delta}_{\boldsymbol{\pi}}\mathbf{\Psi}^{+}(\theta),$$

where $\mathbf{\Delta}_{\boldsymbol{\pi}}$ is the diagonal matrix containing $\boldsymbol{\pi}$.

The inverse problem

- we call a MAP subordinator (H^+, J^+) a $\boldsymbol{\pi}$-friend of (H^-, J^-) if
  1. $\boldsymbol{\Psi} := -\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(-\cdot)\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\Psi}^+$ is a characteristic MAP exponent
  2. $\boldsymbol{\pi}^\top\boldsymbol{\Psi}(0) \leqslant \mathbf{0}^\top$
- then, a MAP (ξ, J) with matrix exponent $\boldsymbol{\Psi}$ is called bonding MAP
- the second condition ensures that
  - $\boldsymbol{\pi}$ is a valid candidate for an invariant distribution of J
  - (H^+, J^+) is a $\boldsymbol{\pi}$-friend of (H^-, J^-) iff (H^-, J^-) is a $\boldsymbol{\pi}$-friend of (H^+, J^+) $\rightsquigarrow$ symmetric relation and the bonding MAP between (H^-, J^-) and (H^+, J^+) is the dual MAP of (ξ, J)

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Questions

1. Are there necessary and sufficient conditions for $\boldsymbol{\pi}$-friendship generalising Vigon's characterisation of Lévy friendships?
2. Is there a concept of MAP philanthropy?

π-compatibility

(H^+, J^+) and (H^-, J^-) are called $\boldsymbol{\pi}$-compatible if

1. if $d_i^{\mp} > 0$, then $\Pi_{i,j}^{\pm}$ restricted to $(0,\infty)$ has a càdlàg density $\partial\Pi_{i,j}^{\pm}$ and $\partial\Pi_{i,i}^{\pm}$ can be expressed as the tail of a signed measure
2. balance conditions on the characteristics that in particular require
   - $q_{i,j}^+ d_i^- F_{i,j}^+(\{0\}) = \frac{\pi(j)}{\pi(i)} q_{j,i}^- d_i^+ F_{j,i}^-(\{0\})$
   - the function

$$x \mapsto q_{i,j}^+ \left(\int_0^\infty \mathbf{1}_{\{y>x\}} \overline{\Pi}_i^-(y-x) F_{i,j}^+(\mathrm{d}y) + d_i^- f_{i,j}^+(x) \right) \\ - \frac{\pi(j)}{\pi(i)} q_{j,i}^- \left(\int_0^\infty \mathbf{1}_{\{-x<y\}} \overline{\Pi}_j^+(x+y) F_{j,i}^-(\mathrm{d}y) + d_j^+ f_{j,i}^-(-x) \right)$$

   is a.e. equal to a right-continuous, bounded variation function converging to 0 at $\pm\infty$
3. the vectors $-\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^+(0)\mathbf{1}$ and $-\boldsymbol{\pi}^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^+(0)$ are nonnegative

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$\boldsymbol{\pi}$-compatibility is necessary for $\boldsymbol{\pi}$-friendship $\rightsquigarrow$ There are no MAP philanthropists!

The Theorem of friends

Theorem (Döring, T. and Watson, 2023+)

(H^+, J^+) and (H^-, J^-) are $\boldsymbol{\pi}$-friends if, and only if, they are $\boldsymbol{\pi}$-compatible and the matrix-valued function

$$\boldsymbol{\Upsilon}(x) = \begin{cases} \Big(\int_{x+}^{\infty} \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \Big(\overline{\boldsymbol{\Pi}}^{-}(y-x) - \boldsymbol{\Psi}^{-}(0) \Big)^{\top} \boldsymbol{\Delta}_{\boldsymbol{\pi}} \, \boldsymbol{\Pi}^{+}(\mathrm{d}y) + \boldsymbol{\Delta}_{\boldsymbol{d}}^{-} \partial \boldsymbol{\Pi}^{+}(x) \Big), & x > 0, \\ \Big(\int_{(-x)+}^{\infty} \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \big(\boldsymbol{\Pi}^{-}(\mathrm{d}y) \big)^{\top} \boldsymbol{\Delta}_{\boldsymbol{\pi}} \left(\overline{\boldsymbol{\Pi}}^{+}(y+x) - \boldsymbol{\Psi}^{+}(0) \right) + \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \big(\boldsymbol{\Delta}_{\boldsymbol{d}}^{+} \partial \boldsymbol{\Pi}^{-}(-x) \big)^{\top} \boldsymbol{\Delta}_{\boldsymbol{\pi}} \Big), & x < 0, \end{cases}$$

is a.e. equal to a function decreasing on $(0, \infty)$ and increasing on $(-\infty, 0)$.

Then, $\boldsymbol{\Upsilon}$ is a.e. the right/left tail of the Lévy measure matrix of the bonding MAP.

Fellowship

We call (H^+, J^+) and (H^-, J^-) $\boldsymbol{\pi}$-fellows if they have decreasing Lévy density matrices $\partial\boldsymbol{\Pi}^+$ and $\partial\boldsymbol{\Pi}^-$ on $(0, \infty)$, and the matrix functions

$$-\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^{-}(0)^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\overline{\boldsymbol{\Pi}}^{+}(x) + \boldsymbol{\Delta}_{\boldsymbol{d}}^{-}\partial\boldsymbol{\Pi}^{+}(x), \quad x > 0,$$

and

$$-\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^{+}(0)^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\overline{\boldsymbol{\Pi}}^{-}(x) + \boldsymbol{\Delta}_{\boldsymbol{d}}^{+}\partial\boldsymbol{\Pi}^{-}(x), \quad x > 0,$$

are decreasing.

Note: Any two Lévy fellows are Lévy philanthropists and therefore friends.

Properties of fellowship

Recall: H^+ Lévy philanthropist $\iff$ H^+ has a decreasing Lévy density $\iff$ H^+ is friends with any pure drift

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Lemma (Döring, T. and Watson, 2023+)

A MAP subordinator (H^+, J^+) with decreasing Lévy density matrix is a $\boldsymbol{\pi}$-friend of a pure drift MAP (H^-, J^-) (that is, $H_t^- = \int_0^t d^-_{J_s^-}\, \mathrm{d}s$) if, and only if, they are $\boldsymbol{\pi}$-compatible $\boldsymbol{\pi}$-fellows.

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Theorem (Döring, T. and Watson, 2023+)

Any two $\boldsymbol{\pi}$-compatible $\boldsymbol{\pi}$-fellows are $\boldsymbol{\pi}$-friends.

Examples

- only known MAP WH-factorisation is from the deep factorisation of the stable process
- to generate explicit friendships it is crucial to find manageable criteria for $\boldsymbol{\pi}$-compatibility
- we develop such criteria in two cases
  1. at least one of the putative friends is a pure drift
  2. both candidates have zero drift parts and no transitional atoms (i.e., $q^{\pm}_{i,j}F^{\pm}_{i,j}(\{0\}) = 0$)
- the first case allows us to give a general construction principle for spectrally one-sided MAPs and modulated Brownian motions starting from the WH-factors
- combining the compatibility criteria from both cases, we construct MAPs jumping in both directions (Markov modulated double exponential jump diffusions)

Probabilistic interpretation of friendships

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- put differently: given a friendship, are the friends (a version) of the ascending/descending ladder height processes of the bonding MAP?

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⇝ verify that Wiener–Hopf factorisation is unique in the sense that for MAP subordinator exponents $\mathbf{\Psi}^{\pm}$ and $\mathbf{\Phi}^{\pm}$ such that

$$\mathbf{\Psi} = -\mathbf{\Delta}_{\pi}^{-1}\mathbf{\Psi}^{-}(-\cdot)^{\top}\mathbf{\Delta}_{\pi}\mathbf{\Psi}^{+} = -\mathbf{\Delta}_{\pi}^{-1}\mathbf{\Phi}^{-}(-\cdot)^{\top}\mathbf{\Delta}_{\pi}\mathbf{\Phi}^{+},$$

it holds

$$\mathbf{\Phi}^{+} = \mathbf{\Delta}\mathbf{\Psi}^{+}, \quad \mathbf{\Phi}^{-} = \mathbf{\Delta}^{-1}\mathbf{\Psi}^{-},$$

for some diagonal matrix $\mathbf{\Delta}$

Uniqueness of the Wiener–Hopf factorisation

- $\mathcal{A}_0$ is the class of matrix exponents of irreducible and finite mean MAP subordinators
- $\mathcal{A}_1$ is the class of MAP subordinator exponents such that for any i one of the following is true
  1. $\lim_{\theta\to\pm\infty}|\psi_i(\theta)| = \infty$
  2. ψ_i is compound Poisson and for any j, $\boldsymbol{\Pi}_{i,j} \ll \text{Leb}$

Theorem (Döring, T. and Watson, 2023+)

Let $(H^\pm, J^\pm)$ be irreducible $\boldsymbol{\pi}$-friends s.t. one of the following sets of conditions holds:

1.
   - the bonding MAP is irreducible and killed,
   - the ladder height processes of the bonding MAP are irreducible, and $\boldsymbol{\pi}$ is invariant for the bonding MAP;
2.
   - the bonding MAP is irreducible and unkilled,
   - $\boldsymbol{\Psi}^\pm \in \mathcal{A}_0 \cap \mathcal{A}_1$, and the exponents of the ladder height processes of the bonding MAP belong to $\mathcal{A}_0 \cap \mathcal{A}_1$.

Then $(H^\pm, J^\pm)$ are versions of the ladder height processes of their bonding MAP.

Summary

- we extend Vigon's theory of Lévy friendships to Markov additive processes
- we demonstrate that there is no direct correspondence of philanthropy in the MAP world
- we coin a different notion, fellowship, that allows us to generate the first explicit MAP WH-factorisations since Kyprianou's deep factorisation of the stable process
- we provide a probabilistic interpretation of friendships by studying uniqueness of the MAP WH-factorisation

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Thank you for your attention!