# Learning to reflect

10th International Conference on Lévy Processes – Mannheim 2022

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joint work with Sören Christensen and Claudia Strauch

18 July 2022

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# Optimal control for Lévy processes

- $\xi$ upward regular Lévy process on $\mathbb{R}$, $\mathbb{E}^0[\xi_1] \in (0, \infty)$
- for impulse controls $S = (\tau_n, \zeta_n)_{n\in\mathbb{N}}$

$$\xi_t^S = \xi_t - \sum_{n:\tau_n \leqslant t} (\xi_{\tau_n,-}^S - \zeta_n)$$

and for a given value function $\gamma$ solve

$$v^* := \sup_S \liminf_{T\to\infty} \frac{1}{T} \mathbb{E}^x \left[ \sum_{n:\tau_n \leqslant T} \left(\gamma\left(\xi_{\tau_n,-}^S\right) - \gamma\left(\zeta_n\right)\right) \right]$$

# Solution for known dynamics

- essential process determining optimal solution: ascending ladder height process $H_t = \xi_{L_t^{-1}}$, where $(L_t)_{t \geqslant 0}$ is local time at supremum of $\xi$
- Reason: for scaling of $L$ s.t. $\mathbb{E}^0[\xi_1] = \mathbb{E}^0[H_1]$ the long term average reward when reflecting in $x$ is given by

$$\mathcal{A}_H \gamma(x) = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}^x[\gamma(\xi_{T_{x+\varepsilon}})] - \gamma(x)}{\mathbb{E}^x[T_{x+\varepsilon}]}, \quad T_y := \inf\{t \geqslant 0 : \xi_t > y\}.$$

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**Theorem** (Christensen, Sohr (2020))

Let $f \coloneqq \mathcal{A}_H \gamma$ be unimodal with maximizer $\theta^*$ (+ technical assumptions). Then $v^* = f(\theta^*)$ and reflecting in $\theta^*$ is optimal.

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3. Analyze sup-norm estimation rates of $\widehat{f}_T$ to determine regret of the strategy, since for $\theta^* \in K$

$$\mathbb{E}^0[f(\theta^*) - f(\widehat{\theta}_T)] \leqslant 2\mathbb{E}^0[\|\widehat{f}_T - f\|_{L^\infty(K^\circ)}].$$

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**Statistical challenge**

How can we build an estimator of $\mathcal{A}_H \gamma$ although local time $L$ cannot be observed?

# Construction of a spatial estimator

- Integration by parts reveals

$$\mathcal{A}_H\gamma(x) = d_H\gamma'(x) + \int_{0+}^{\infty} (\gamma(x+y) - \gamma(x))\, \Pi_H(\mathrm{d}y) = \int_0^{\infty} \eta\gamma'(x+y)\, \mu(\mathrm{d}y),$$

where $\eta = \mathbb{E}^0[\xi_1]$ and

$$\mu(\mathrm{d}y) = \frac{1}{\mathbb{E}^0[\xi_1]}\Big(d_H\delta_0(\mathrm{d}y) + \Pi_H((y,\infty))\,\mathrm{d}y\mathbf{1}_{(0,\infty)}(y)\Big), \quad y \geqslant 0.$$

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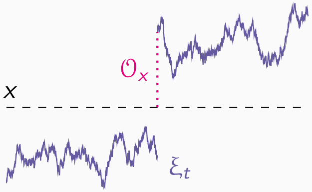
- Let

$$\mathcal{O}_x = \xi_{T_x} - x, \quad x \geqslant 0,$$

be the overshoot of $\xi$ over a level $x$.



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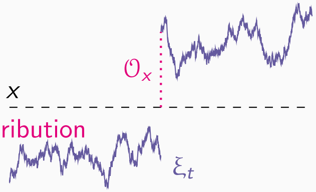

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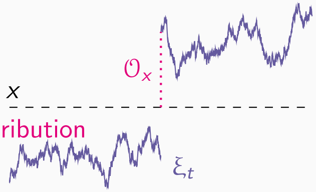


- $(\mathcal{O}_x)_{x\geqslant 0}$ is a $\mathbb{R}_+$-valued Feller process and $\mu$ is its invariant distribution
- natural (spatial) estimator

$$\widetilde{f}_Y(x) = \frac{1}{Y}\int_0^Y \eta\gamma'(x+\mathcal{O}_y)\,\mathrm{d}y,$$

given data $(\xi_{T_y})_{y\in[0,Y]}$.

Note that

$$\mathbb{E}^0\left[\|\widetilde{f}_Y - f\|_{L^\infty(D)}\right] = \mathbb{E}^0\left[\sup_{g\in\mathcal{G}}\left|\frac{1}{Y}\int_0^Y g(\mathcal{O}_y)\,\mathrm{d}y\right|\right],$$

where

$$\mathcal{G} = \{\eta\gamma'(x+\cdot) - \mu(\eta\gamma'(x+\cdot)) : x \in D \cap \mathbb{Q}\}$$

# From overshoots to path integrals of Markov processes

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**Question**

Which stability properties are needed to obtain useful quantitative uniform moment bounds on Markovian path integrals $\int_0^T g(X_t)\,\mathrm{d}t$ for $\mu$-centered functions $g$?

# Uniform moment bounds

- $X$ stationary $\mathcal{X}$-valued Borel right Markov process with invariant distribution $\mu$
- $X$ is exponentially $\beta$-mixing, i.e.,

$$\beta(t) \coloneqq \int_{\mathcal{X}} \|\mathbb{P}^x(X_t \in \cdot) - \mu\|_{\mathrm{TV}}\, \mu(\mathrm{d}x) \lesssim \mathrm{e}^{-\kappa t}, \quad t > 0.$$

- exponential $\beta$-mixing is implied by exponential ergodicity,

$$\|\mathbb{P}^x(X_t \in \cdot) - \mu\|_{\mathrm{TV}} \lesssim V(x)\mathrm{e}^{-\kappa t}, \quad x \in \mathcal{X}, t > 0.$$

- for countable family $\mathcal{G} \subset \mathcal{B}_b(\mathcal{X})$ of $\mu$-centered functions let

$$\mathbb{G}_T(g) \coloneqq \frac{1}{\sqrt{T}} \int_0^T g(X_s)\, \mathrm{d}s, \quad g \in \mathcal{G}$$

  and

$$d_{\mathbb{G},T}(f,g) \coloneqq \sqrt{\mathrm{Var}(\mathbb{G}_T(f-g))}$$

**Theorem** (Dexheimer, Strauch, T. (2022+))

Let $X$ be exp. $\beta$-mixing and $m_T \leqslant T/4$. Then, there exists $\tau \in [m_T, 2m_T]$ such that for any $p \geqslant 1$,

$$\left(\mathbb{E}^{\mu}\left[\sup_{g\in\mathcal{G}} |\mathbb{G}_T(g)|^p\right]\right)^{1/p} \leqslant C_1 \int_0^\infty \log \mathcal{N}\big(u, \mathcal{G}, \tfrac{2m_T}{\sqrt{T}} d_\infty\big)\, \mathrm{d}u + C_2 \int_0^\infty \sqrt{\log \mathcal{N}(u, \mathcal{G}, d_{\mathbb{G},\tau})}\, \mathrm{d}u$$
$$+ 4 \sup_{g\in\mathcal{G}} \left(\frac{2m_T}{\sqrt{T}} \|g\|_\infty c_1 p + \|g\|_{\mathbb{G},\tau} c_2 \sqrt{p} + \frac{1}{2}\|g\|_\infty c_\kappa \sqrt{T} \mathrm{e}^{-\frac{\kappa m_T}{p}}\right),$$

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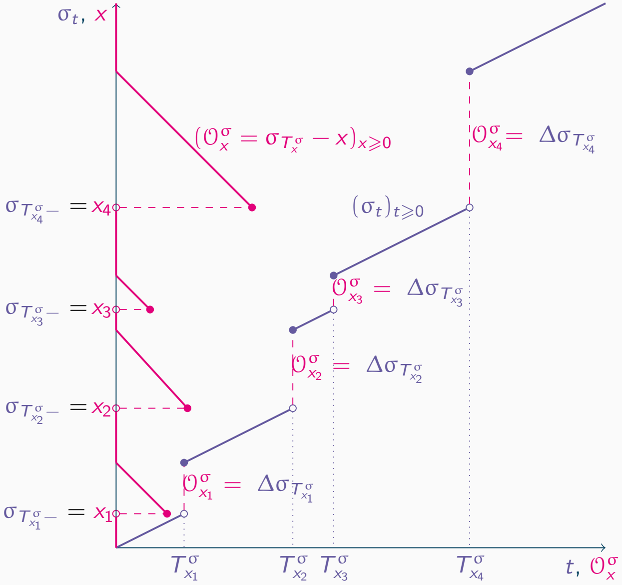
**Consequence**

**IF** $\mathcal{O}$ is exponentially ergodic and $\gamma', \gamma''$ are bounded, then $\mathbb{E}^0\left[\|\widetilde{f}_Y - f\|_{L^\infty(D)}\right] \lesssim \frac{1}{\sqrt{Y}}$.



$\sigma_t,\ x$
$(\mathcal{O}^\sigma_x = \sigma_{T^\sigma_x} - x)_{x\geqslant 0}$
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$T^\sigma_{x_1}$
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$T^\sigma_{x_4}$
$t,\ \mathcal{O}^\sigma_x$

# Stability of overshoots

**Theorem** (Döring, T. (2021+))

Assume that $\mathbb{E}^0[H_1] < \infty$ and either $d_H > 0$ or $\Pi|_{(a,b)} \ll \mathrm{Leb}|_{(a,b)}$ for some interval $(a, b) \subset (0, \infty)$. Then, for any $y \in \mathbb{R}_+$,

$$\lim_{x\to\infty} \left\|\mathbb{P}^y(\mathcal{O}_x \in \cdot) - \mu\right\|_{\mathrm{TV}} = 0.$$

If additionally for some $\lambda > 0$, $\int_1^\infty e^{\lambda x}\,\Pi(dx) < \infty$, then for $V_\lambda(x) = \exp(\lambda x)$ and $\alpha > 0$, we have

$$\sup_{y\in\mathbb{R}_+} \frac{\sup_{|f|\leqslant \alpha\mathcal{U}_\alpha V_\lambda} \left|\mathbb{E}^y[f(\mathcal{O}_x)] - \mu(f)\right|}{\alpha\mathcal{U}_\alpha V_\lambda(y)} \lesssim_\alpha e^{-\kappa(\alpha)x},$$

and for any $\delta \in (0, 1)$,

$$\left\|\mathbb{P}^y(\mathcal{O}_x \in \cdot) - \mu\right\|_{\mathrm{TV}} \lesssim_{(\lambda,\delta)} \mathcal{U}_{2\lambda/\delta} V_\lambda(y) e^{-\frac{\lambda x}{1+\delta}}.$$

Moreover, under the above assumptions, $\mathcal{O}$ is exponentially $\beta$-mixing for any initial distribution $\eta$ such that $\eta(V_\lambda) < \infty$.

# Back to data-driven Lévy controls

- Recall: for $f = \mathcal{A}_H \gamma$ and $\widetilde{f}_Y(x) = \frac{1}{Y}\int_0^Y \eta\gamma'(x + \mathcal{O}_y)\,\mathrm{d}y$ we have $\mathbb{E}^0[\|\widetilde{f}_Y - f\|_{L^\infty(D)}] \lesssim Y^{-1/2}$ in the exponentially ergodic overshoot regime

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- $\mathbb{E}^0\big[\big\|\widehat{f}_T - f\big\|_{L^\infty(D)}\big] \lesssim \frac{1}{\sqrt{\mathbb{E}^0[\xi_1]T}} + \frac{\varepsilon}{\mathbb{E}^0[\xi_1]} + \mathbb{P}^0\Big(\Big|\frac{\xi_T}{T} - \mathbb{E}^0[\xi_1]\Big| > \varepsilon\Big)$

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**Theorem** (Christensen, Strauch, T. (2021+))

Assume $\theta^* \in D$ and let $\widehat{\theta}_T = \arg\max_{x\in D} \widehat{f}_T(x)$. Given exponentially ergodic overshoots,

$$\mathbb{E}^0\big[f(\theta^*) - f(\widehat{\theta}_T)\big] \in \mathrm{O}\big(T^{-1/(2(1+p^{-1}))}\big),$$

provided that $\int_{|x|>1} |x|^p\,\Pi(\mathrm{d}x) < \infty$ for some $p \geqslant 2$. If $\Pi$ has an exponential moment, then

$$\mathbb{E}^0\big[f(\theta^*) - f(\widehat{\theta}_T)\big] \in \mathrm{O}\Big(\sqrt{\log T/T}\Big).$$

# Final Remarks

- no exploration-exploitation tradeoff due to spatial homogeneity of $\xi$
- it is an interesting problem to extend results to a high-frequency setting based on random walk observations $(\xi_{k\Delta_n})_{k=0,\dots,n}$ for $n\Delta_n \to \infty$
- For an oscillating Lévy process with ascending/descending ladder height processes $H^{\pm}$ we have

$$\mathcal{A}^* f = (\mu^+ * \mu^-)'' * f = (\mu^+ * \mu^-) * f'', \quad f \in \mathcal{S}(\mathbb{R})$$

where $\mu^+$ (resp. $\mu^-(-\cdot)$) is the invariant overshoot measure of $H^+$ (resp. $H^-$) $\rightsquigarrow$ can this be used for convenient constructions of friendships of Lévy processes?

Thank you for your attention!