Learning to reflect

ISNPS – Paphos 2022

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23 June 2022

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Optimal control for Lévy processes

- ξ upward regular Lévy process on $\mathbb{R}$, $\mathbb{E}^0[\xi_1] \in (0, \infty)$
- for impulse controls $S = (\tau_n, \zeta_n)_{n\in\mathbb{N}}$

$$\xi_t^S = \xi_t - \sum_{n:\tau_n \leqslant t} (\xi^S_{\tau_n,-} - \zeta_n)$$

and for a given value function γ solve

$$v^* \coloneqq \sup_S \liminf_{T\to\infty} \frac{1}{T} \mathbb{E}^x \Bigg[\sum_{n:\tau_n \leqslant T} \left(\gamma\left(\xi^S_{\tau_n,-}\right) - \gamma\left(\zeta_n\right)\right) \Bigg]$$

Solution for known dynamics

- essential process determining optimal solution: ascending ladder height process $H_t = \xi_{L_t^{-1}}$, where $(L_t)_{t \geqslant 0}$ is local time at supremum of ξ
- Reason: for scaling of L s.t. $\mathbb{E}^0[\xi_1] = \mathbb{E}^0[H_1]$ the long term average reward when reflecting in x is given by

$$\mathcal{A}_H \gamma(x) = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}^x[\gamma(\xi_{T_{x+\varepsilon}})] - \gamma(x)}{\mathbb{E}^x[T_{x+\varepsilon}]}, \quad T_y \coloneqq \inf\{t \geqslant 0 : \xi_t > y\}.$$

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Theorem (Christensen, Sohr (2020))

Let $f \coloneqq \mathcal{A}_H\gamma$ be unimodal with maximizer θ^* (+ technical assumptions). Then $v^* = f(\theta^*)$ and reflecting in θ^* is optimal.

Data-driven strategy

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3. Analyze sup-norm estimation rates of $\widehat{f}_T$ to determine regret of the strategy, since for $\theta^* \in K$

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Statistical challenge

How can we build an estimator of $\mathcal{A}_H\gamma$ although local time L cannot be observed?

Construction of a spatial estimator

- Integration by parts reveals

$$\mathcal{A}_H\gamma(x) = d_H\gamma'(x) + \int_{0+}^{\infty} (\gamma(x+y) - \gamma(x))\,\Pi_H(\mathrm{d}y) = \int_0^{\infty} \eta\gamma'(x+y)\,\mu(\mathrm{d}y),$$

where $\eta = \mathbb{E}^0[\xi_1]$ and

$$\mu(\mathrm{d}y) = \frac{1}{\mathbb{E}^0[\xi_1]}\Big(d_H\delta_0(\mathrm{d}y) + \Pi_H((y,\infty))\,\mathrm{d}y\mathbf{1}_{(0,\infty)}(y)\Big), \quad y \geqslant 0.$$

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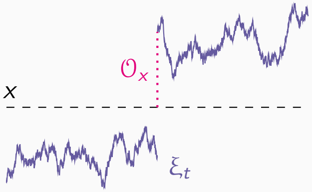
- Let

$$\mathcal{O}_x = \xi_{T_x} - x, \quad x \geqslant 0,$$

be the overshoot of ξ over a level x.



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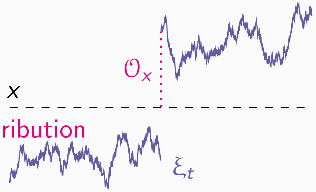

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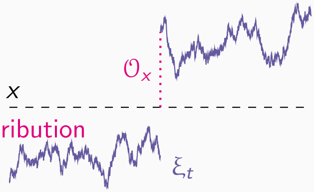

- $(\mathcal{O}_x)_{x\geqslant 0}$ is a $\mathbb{R}_+$-valued Feller process and μ is its invariant distribution
- natural (spatial) estimator

$$\widetilde{f}_Y(x) = \frac{1}{Y}\int_0^Y \eta\gamma'(x + \mathcal{O}_y)\,\mathrm{d}y,$$

  given data $(\xi_{T_y})_{y\in[0,Y]}$.

From overshoots to path integrals of Markov processes

Note that

$$\mathbb{E}^0\left[\|\widetilde{f}_Y - f\|_{L^\infty(D)}\right] = \mathbb{E}^0\left[\sup_{g\in\mathcal{G}}\left|\frac{1}{Y}\int_0^Y g(\mathcal{O}_y)\,\mathrm{d}y\right|\right],$$

where

$$\mathcal{G} = \{\eta\gamma'(x+\cdot) - \mu(\eta\gamma'(x+\cdot)) : x \in D \cap \mathbb{Q}\}$$

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Question

Which stability properties are needed to obtain useful quantitative uniform moment bounds on Markovian path integrals $\int_0^T g(X_t)\,\mathrm{d}t$ for μ-centered functions g?

Uniform moment bounds

- X stationary $\mathcal{X}$-valued Borel right Markov process with invariant distribution μ
- X is exponentially β-mixing, i.e.,

$$\beta(t) \coloneqq \int_{\mathcal{X}} \|\mathbb{P}^x(X_t \in \cdot) - \mu\|_{\mathrm{TV}}\, \mu(\mathrm{d}x) \lesssim \mathrm{e}^{-\kappa t}, \quad t > 0.$$

- exponential β-mixing is implied by exponential ergodicity,

$$\|\mathbb{P}^x(X_t \in \cdot) - \mu\|_{\mathrm{TV}} \lesssim V(x)\mathrm{e}^{-\kappa t}, \quad x \in \mathcal{X}, t > 0.$$

- for countable family $\mathcal{G} \subset \mathcal{B}_b(\mathcal{X})$ of μ-centered functions let

$$\mathbb{G}_T(g) \coloneqq \frac{1}{\sqrt{T}} \int_0^T g(X_s)\,\mathrm{d}s, \quad g \in \mathcal{G}$$

  and

$$d_{\mathbb{G},T}(f,g) \coloneqq \sqrt{\mathrm{Var}(\mathbb{G}_T(f-g))}$$

Theorem (Dexheimer, Strauch, T. (2022+))

Let X be exp. β-mixing and $m_T \leqslant T/4$. Then, there exists $\tau \in [m_T, 2m_T]$ such that for any $p \geqslant 1$,

$$\left(\mathbb{E}^{\mu}\left[\sup_{g\in\mathcal{G}} |\mathbb{G}_T(g)|^p\right]\right)^{1/p} \leqslant C_1 \int_0^{\infty} \log \mathcal{N}\big(u, \mathcal{G}, \tfrac{2m_T}{\sqrt{T}} d_\infty\big)\,\mathrm{d}u + C_2 \int_0^{\infty} \sqrt{\log \mathcal{N}(u, \mathcal{G}, d_{\mathbb{G},\tau})}\,\mathrm{d}u$$
$$+ 4 \sup_{g\in\mathcal{G}} \left(\frac{2m_T}{\sqrt{T}} \|g\|_\infty c_1 p + \|g\|_{\mathbb{G},\tau} c_2 \sqrt{p} + \frac{1}{2} \|g\|_\infty c_\kappa \sqrt{T} \mathrm{e}^{-\frac{\kappa m_T}{p}}\right),$$

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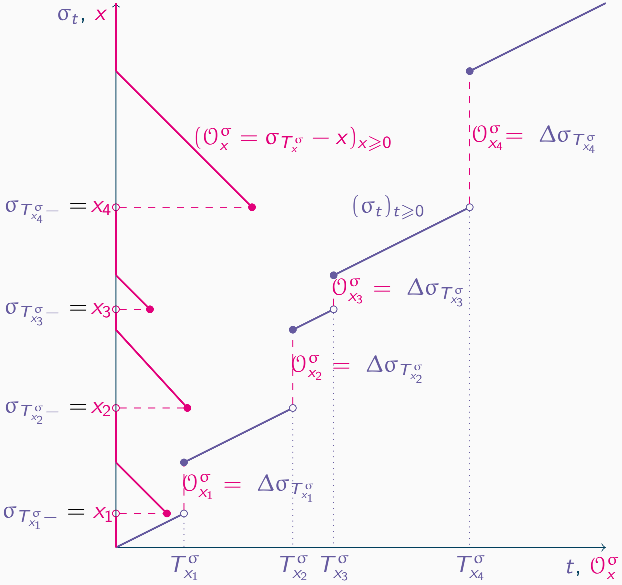
Consequence

IF $\mathcal{O}$ is exponentially ergodic and γ', γ'' are bounded, then $\mathbb{E}^0\big[\|\widetilde{f}_Y - f\|_{L^\infty(D)}\big] \lesssim \frac{1}{\sqrt{Y}}$.


$\sigma_t,\ x$
$(\mathcal{O}^\sigma_x = \sigma_{T^\sigma_x} - x)_{x\geqslant 0}$
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$t,\ \mathcal{O}^\sigma_x$

Stability of overshoots

Theorem (Döring, T. (2021+))

Assume that $\mathbb{E}^0[H_1] < \infty$ and either $d_H > 0$ or $\Pi|_{(a,b)} \ll \mathrm{Leb}|_{(a,b)}$ for some interval $(a, b) \subset (0, \infty)$. Then, for any $y \in \mathbb{R}_+$,

$$\lim_{x \to \infty} \left\| \mathbb{P}^y(\mathcal{O}_x \in \cdot) - \mu \right\|_{\mathrm{TV}} = 0.$$

If additionally for some $\lambda > 0$, $\int_1^\infty \mathrm{e}^{\lambda x}\,\Pi(\mathrm{d}x) < \infty$, then for $V_\lambda(x) = \exp(\lambda x)$ and $\alpha > 0$, we have

$$\sup_{y \in \mathbb{R}_+} \frac{\sup_{|f| \leqslant \alpha \mathcal{U}_\alpha V_\lambda} \left| \mathbb{E}^y[f(\mathcal{O}_x)] - \mu(f) \right|}{\alpha \mathcal{U}_\alpha V_\lambda(y)} \lesssim_\alpha \mathrm{e}^{-\kappa(\alpha) x},$$

and for any $\delta \in (0, 1)$,

$$\left\| \mathbb{P}^y(\mathcal{O}_x \in \cdot) - \mu \right\|_{\mathrm{TV}} \lesssim_{(\lambda,\delta)} \mathcal{U}_{2\lambda/\delta} V_\lambda(y) \mathrm{e}^{-\frac{\lambda x}{1+\delta}}.$$

Moreover, under the above assumptions, $\mathcal{O}$ is exponentially β-mixing for any initial distribution η such that $\eta(V_\lambda) < \infty$.

Back to data-driven Lévy controls

- Recall: for $f = \mathcal{A}_H \gamma$ and $\widetilde{f}_Y(x) = \frac{1}{Y}\int_0^Y \eta\gamma'(x + \mathcal{O}_y)\,\mathrm{d}y$ we have $\mathbb{E}^0[\|\widetilde{f}_Y - f\|_{L^\infty(D)}] \lesssim Y^{-1/2}$ in the exponentially ergodic overshoot regime

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- $\mathbb{E}^0\big[\big\|\widehat{f}_T - f\big\|_{L^\infty(D)}\big] \lesssim \frac{1}{\sqrt{\mathbb{E}^0[\xi_1]T}} + \frac{\varepsilon}{\mathbb{E}^0[\xi_1]} + \mathbb{P}^0\big(\big|\frac{\xi_T}{T} - \mathbb{E}^0[\xi_1]\big| > \varepsilon\big)$

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Theorem (Christensen, Strauch, T. (2021+))

Assume $\theta^* \in D$ and let $\widehat{\theta}_T = \arg\max_{x\in D} \widehat{f}_T(x)$. Given exponentially ergodic overshoots,

$$\mathbb{E}^0\big[f(\theta^*) - f(\widehat{\theta}_T)\big] \in \mathrm{O}\big(T^{-1/(2(1+p^{-1}))}\big),$$

provided that $\int_{|x|>1}|x|^p\,\Pi(\mathrm{d}x) < \infty$ for some $p \geqslant 2$. If Π has an exponential moment, then

$$\mathbb{E}^0\big[f(\theta^*) - f(\widehat{\theta}_T)\big] \in \mathrm{O}\Big(\sqrt{\log T/T}\Big).$$

Final Remarks

- no exploration-exploitation tradeoff due to spatial homogeneity of ξ
- it is an interesting problem to extend results to a high-frequency setting based on random walk observations $(\xi_{k\Delta_n})_{k=0,\dots,n}$ for $n\Delta_n \to \infty$
- For an oscillating Lévy process with ascending/descending ladder height processes $H^{\pm}$ we have

$$\mathcal{A}^* f = (\mu^+ * \mu^-)'' * f = (\mu^+ * \mu^-) * f'', \quad f \in \mathcal{S}(\mathbb{R})$$

  where μ^+ (resp. $\mu^-(-\cdot)$) is the invariant overshoot measure of H^+ (resp. H^-) $\rightsquigarrow$ can this be used for convenient constructions of friendships of Lévy processes?

Thank you for your attention!