On friendships of Lévy and Markov additive processes

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Theory of friends for Lévy processes

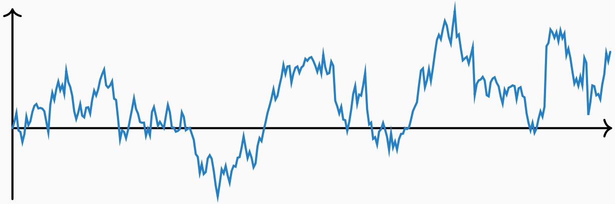
Lévy processes

- a (killed) Lévy process ξ is a càdlàg process with stationary, independent increments
- characteristic exponent ψ characterised via $\mathbb{E}[e^{i\theta\xi_t}] = e^{t\psi(\theta)}$
- Lévy–Khintchine formula:

$$\psi(\theta) = -\dagger + iax - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x \mathbf{1}_{[-1,1]}(x))\,\Pi(dx), \quad \theta \in \mathbb{R}$$

- $\dagger$ is the killing rate, $a \in \mathbb{R}$ is the centre, $\sigma \geq 0$ the Gaussian coefficient, Π is the Lévy measure that controls size and frequency of jumps of the process
- (a, σ, Π) is called the characteristic triplet of ψ

Wiener–Hopf factorisation (path picture)



ξ, a Lévy process

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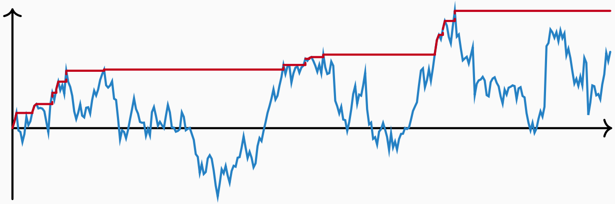

- ξ, a Lévy process
- $\bar{\xi}_t = \sup_{s \le t} \xi_s$, the running supremum

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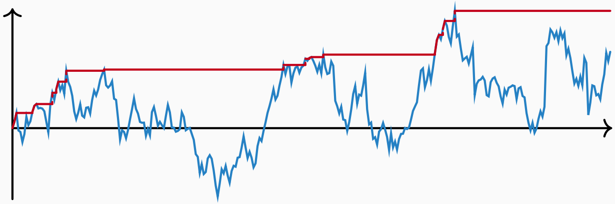

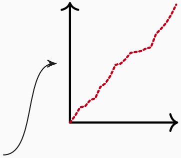

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H_t^+, the ascending ladder height process: suprema ‘stitched together’

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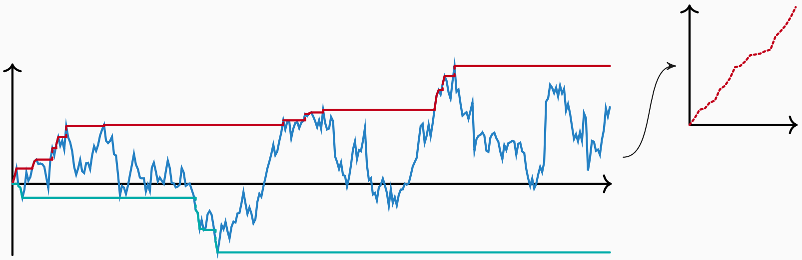

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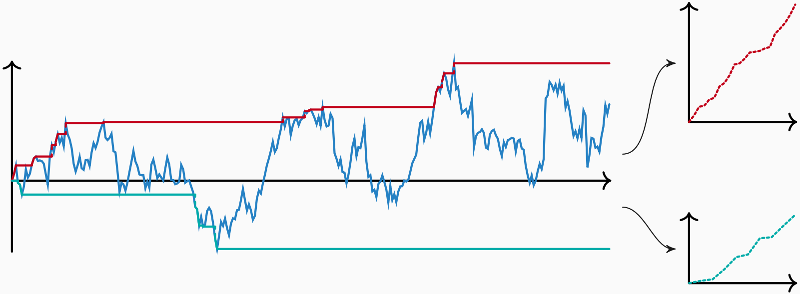

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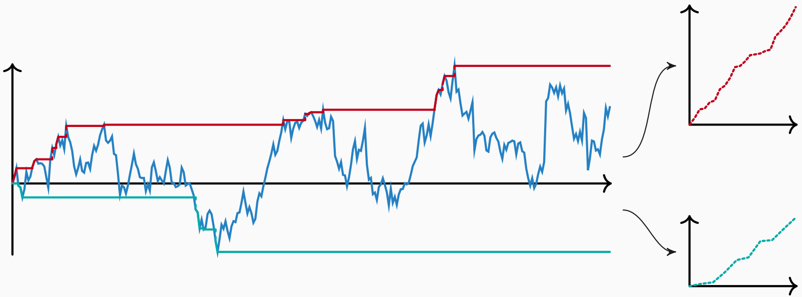

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$H^{\pm}$ are subordinators (increasing Lévy processes).

Spatial Wiener–Hopf factorisation

Let $\psi^{\pm}$ denote the characteristic exponents of $H^{\pm}$ (for some specific scaling of local times at the supremum and infimum)

Theorem

(i) For an independent Exp(q)-distributed random time $\boldsymbol{e}_q$ it holds that $\bar{\xi}_{\boldsymbol{e}_q}$ and $\bar{\xi}_{\boldsymbol{e}_q} - \xi_{\boldsymbol{e}_q}$ are infinitely divisible and independent, where $\bar{\xi}_{\boldsymbol{e}_q} - \xi_{\boldsymbol{e}_q} \overset{d}{=} -\underline{\xi}_{\boldsymbol{e}_q}$.

(ii) For appropriate scalings of local times at the supremum and infimum, it holds that

$$\psi(\theta) = -\psi^{-}(-\theta)\psi^{+}(\theta), \quad \theta \in \mathbb{R}.$$

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Problem

Given a LK exponent ψ, only in rare special cases can the Wiener–Hopf factors be explicitly determined.

The inverse problem

- start with two subordinators $H^{\pm}$ having LK exponents $\psi^{\pm}$
- is there a Lévy process ξ with LK exponent ψ such that $\psi = -\psi^{-}(-\cdot)\psi^{+}$?
- when such ξ exists, we call $H^{\pm}$ friends and ξ the bonding process

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Vigon's powerful point of view

For $\varphi \in \mathcal{S}(\mathbb{R})$ let

$$\langle \mathbb{\Gamma}^2\Pi, \varphi\rangle := \int_{\mathbb{R}} (\varphi(x) - \varphi(0) - \varphi'(x)\mathbf{1}_{[-1,1]}(x))\,\Pi(\mathrm{d}x), \quad \langle \mathbb{\Gamma}\Pi^{\pm}, \varphi\rangle := \int_{\mathbb{R}} (\varphi(x) - \varphi(0))\,\Pi^{\pm}(\mathrm{d}x).$$

Then, in the sense of tempered distributions

$$\psi = \mathscr{F}\Big\{\underbrace{-\dagger\delta - a\delta' + \frac{\sigma^2}{2}\delta'' + \mathbb{\Gamma}^2\Pi}_{=:G}\Big\}, \quad \psi^{\pm} = \mathscr{F}\Big\{\underbrace{-\dagger^{\pm}\delta - d^{\pm}\delta' + \mathbb{\Gamma}\Pi^{\pm}}_{=:G^{\pm}}\Big\},$$

and the Wiener–Hopf factorisation becomes the convolution equality

$$G = -\widetilde{G}^{-} * G^{+}.$$

The Theorem of friends

- $d^{\pm}$ drift of $H^{\pm}$
- $\Pi^{\pm}$ Lévy measure of $H^{\pm}$
- $H^{\pm}$ are compatible if $d^{\mp} > 0$ implies that $\Pi^{\pm}$ has a càdlàg density $\partial\Pi^{\pm}$ that can be expressed as the tail of a signed measure

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Theorem (Vigon, 2002)[1,2]

H^+ and H^- are friends if, and only if, they are compatible and the function

$$\Upsilon(x) = \begin{cases} \int_{x+}^{\infty} (\Pi^-(y-x,\infty) - \psi^-(0))\,\Pi^+(\mathrm{d}y) + d^-\partial\Pi^+(x), & x > 0, \\ \int_{(-x)+}^{\infty} (\Pi^+(y+x,\infty) - \psi^+(0))\,\Pi^-(\mathrm{d}y) + d^+\partial\Pi^-(-x), & x < 0, \end{cases}$$

is a.e. increasing on $(0,\infty)$ and a.e. decreasing on $(-\infty, 0)$.

If they are friends, then Υ is a.e. the right/left tail of the Lévy measure of the bonding process.

[1] V. Vigon (2002). Votre Lévy rampe-t-il? *J. Lond. Math. Soc.*
[2] V. Vigon (2002). Simplifiez vos Lévy en titillant la factorisation de Wiener-Hopf. *PhD Thesis.*

Philanthropy

- a subordinator H^+ is called philanthropist if its Lévy measure admits a decreasing density

$\Longleftrightarrow$ philanthropists are subordinators that are friends with pure drift subordinators $H_t^- = d^- t$

$\rightsquigarrow$ spectrally negative Lévy processes that do not drift to $-\infty$ can be factorised into a philanthropist and a pure drift

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Example (Kuznetsov, Kyprianou, Pardo, van Schaik, 2011)[3]

For $\beta_\pm \geq 0, \gamma_\pm \in (0,1)$, the exponents

$$\psi^\pm(\theta) = \frac{\Gamma(\beta_\pm + \gamma_\pm - \mathrm{i}\theta)}{\Gamma(\beta_\pm - \mathrm{i}\theta)},$$

are philanthropists and their bonding process is called a hypergeometric Lévy process.

[3] Kuznetsov, A., Kyprianou, A. E., Pardo, J. C. and K. van Schaik (2011). A Wiener-Hopf Monte-Carlo simulation technique for Lévy processes. *Ann. Appl. Probab*

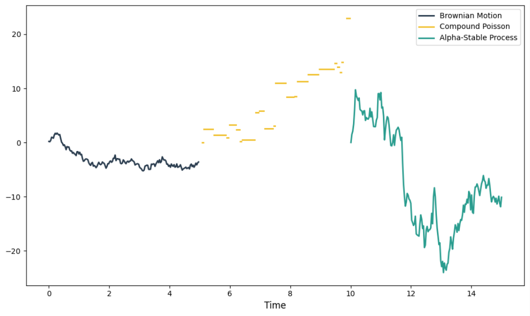
Markov additive friendships

Markov additive processes

- a Feller process (ξ, J) is a MAP on $\mathbb{R} \times \{1, \dots, n\}$ if

$$\mathbb{E}^{0,i}[f(\xi_{t+s} - \xi_t, J_{t+s}) \mid \mathcal{F}_t]\mathbf{1}_{\{t<\zeta\}} = \mathbb{E}^{0,J_t}[f(\xi_s, J_s)]\mathbf{1}_{\{t<\zeta\}}$$

- equivalently, a MAP can be characterised as a regime-switching Lévy process
    - $\xi^{(i)}$ is a Lévy process for any phase $i \in [n]$
    - J is a Markov chain with transition matrix Q
    - when J is in state i, run an independent copy of $\xi^{(i)}$
    - phase switches from i to j trigger an additional jump with distribution $F_{i,j}$



Analytic characterisation of MAPs

- ψ_i LK exponent of $\xi^{(i)}$
- Π_i Lévy measure of $\xi^{(i)}$ and denote $\Pi_{i,j} := q_{i,j} F_{i,j}$
- call $\boldsymbol{\Pi} = (\Pi_{i,j})_{i,j\in[n]}$ Lévy measure matrix of ξ
- we have $\mathbb{E}^{0,i}[\mathrm{e}^{\mathrm{i}\theta\xi_t}\mathbf{1}_{\{J_t=j\}}] = (\exp(t\boldsymbol{\Psi}(\theta)))_{i,j}$ with characteristic matrix exponent

$$\boldsymbol{\Psi}(\theta) = \mathrm{diag}\big((\psi_i(\theta))_{i\in[n]}\big) + \boldsymbol{Q} \odot \big(\widehat{F_{i,j}}(\theta)\big)_{i,j\in[n]}$$

$$= \begin{bmatrix} \psi_1(\theta) + q_{1,1} & \widehat{\Pi}_{1,2}(\theta) & \cdots & \widehat{\Pi}_{1,n}(\theta) \\ \widehat{\Pi}_{2,1}(\theta) & \psi_2(\theta) + q_{2,2} & \cdots & \widehat{\Pi}_{2,n}(\theta) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\Pi}_{n,1}(\theta) & \widehat{\Pi}_{n,2}(\theta) & \cdots & \psi_n(\theta) + q_{n,n} \end{bmatrix}$$

MAP Wiener–Hopf factorisation

- ascending/descending ladder height MAPs $(H^{\pm}, J^{\pm})$: ordinator $H^{\pm}$ tracks new suprema/infima of ξ and $J^{\pm}$ tracks phases during which they occur
- they are MAP subordinators (increasing ordinators) with matrix exponents $\boldsymbol{\Psi}^{\pm}$
- we always assume that J is irreducible and hence has an invariant distribution represented by a vector $\boldsymbol{\pi}$

Theorem (Dereich, Döring, Kyprianou (2017)[4]; Ivanovs (2017)[5]; Döring, T., Watson (2024)[6])

$$\boldsymbol{\Psi}(\theta) = -\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^{-}(-\theta)^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\Psi}^{+}(\theta),$$

where $\boldsymbol{\Delta}_{\boldsymbol{\pi}}$ is the diagonal matrix containing $\boldsymbol{\pi}$.

[4] S. Dereich, L. Döring and A.E. Kyprianou (2017). Reals self-similar processes started from the origin. *Ann. Probab.*
[5] J. Ivanovs (2017). Splitting and time-reversal for Markov additive processes. *Stochastic Process. Appl.*
[6] L. Döring, L. Trottner and A.R. Watson (2024). Markov additive friendships. *Trans. Amer. Math. Soc.*

The inverse problem

- we call a MAP subordinator (H^+, J^+) a $\boldsymbol{\pi}$-friend of (H^-, J^-) if
  1. $\boldsymbol{\Psi} := -\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(-\cdot)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\Psi}^+$ is a characteristic MAP exponent
  2. $\boldsymbol{\pi}^\top \boldsymbol{\Psi}(0) \le \mathbf{0}^\top$
- then, a MAP (ξ, J) with matrix exponent $\boldsymbol{\Psi}$ is called bonding MAP
- the second condition ensures that
  - $\boldsymbol{\pi}$ is a valid candidate for an invariant distribution of J
  - (H^+, J^+) is a $\boldsymbol{\pi}$-friend of (H^-, J^-) iff (H^-, J^-) is a $\boldsymbol{\pi}$-friend of (H^+, J^+) $\rightsquigarrow$ symmetric relation and the bonding MAP between (H^-, J^-) and (H^+, J^+) is the dual MAP of (ξ, J)

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Questions

1. Are there necessary and sufficient conditions for $\boldsymbol{\pi}$-friendship generalising Vigon's characterisation of Lévy friendships?
2. Is there a concept of MAP philanthropy?

π-compatibility

(H^+, J^+) and (H^-, J^-) are called $\boldsymbol{\pi}$-compatible if

1. $d_i^{\mp} > 0$, then $\Pi_{i,j}^{\pm}$ has a càdlàg density $\partial\Pi_{i,j}^{\pm}$ on $(0,\infty)$ and $\partial\Pi_{i,j}^{\pm}$ can be expressed as the tail of a signed measure;
2. balance conditions on the characteristics are fulfilled that in particular require
   - $q_{i,j}^+ d_i^- F_{i,j}^+(\{0\}) = \frac{\pi(j)}{\pi(i)} q_{j,i}^- d_i^+ F_{j,i}^-(\{0\})$
   - the function

$$x \mapsto q_{i,j}^+ \underbrace{\left(\int_0^\infty \mathbf{1}_{\{y>x\}} \overline{\Pi}_i^-(y-x) F_{i,j}^+(\mathrm{d}y) + d_i^- f_{i,j}^+(x) \right)}_{=\tilde{\chi}_i^- * F_{i,j}^+(\mathrm{d}x)/\,\mathrm{d}x} - \frac{\pi(j)}{\pi(i)} q_{j,i}^- \underbrace{\left(\int_0^\infty \mathbf{1}_{\{-x<y\}} \overline{\Pi}_j^+(x+y)\, F_{j,i}^-(\mathrm{d}y) + d_j^+ f_{j,i}^-(-x) \right)}_{=\chi_j^+ * \widetilde{F}_{j,i}^-(\mathrm{d}x)/\,\mathrm{d}x}$$

   is a.e. equal to a right-continuous function of bounded variation that converges to 0 at $\pm\infty$. Above,

$$\chi_i^{\pm}(\mathrm{d}x) = d_i^{\pm}\delta_0(\mathrm{d}x) + \mathbf{1}_{(0,\infty)}(x)\overline{\Pi}_i^{\pm}(x)\,\mathrm{d}x,$$

   denotes the invariant overshoot measure of $H^{\pm,(i)}$;
3. the vectors $-\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^+(0)\mathbf{1}$ and $-\boldsymbol{\pi}^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Psi}^+(0)$ are nonnegative.

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   - $q_{i,j}^{+} d_i^{-} F_{i,j}^{+}(\{0\}) = \frac{\pi(j)}{\pi(i)} q_{j,i}^{-} d_i^{+} F_{j,i}^{-}(\{0\})$
   - the function

$$x \mapsto q_{i,j}^{+} \underbrace{\left(\int_0^\infty \mathbf{1}_{\{y>x\}} \overline{\Pi}_i^{-}(y-x) F_{i,j}^{+}(\mathrm{d}y) + d_i^{-} f_{i,j}^{+}(x) \right)}_{=\tilde{\chi}_i^{-} * F_{i,j}^{+}(\mathrm{d}x)/\,\mathrm{d}x} - \frac{\pi(j)}{\pi(i)} q_{j,i}^{-} \underbrace{\left(\int_0^\infty \mathbf{1}_{\{-x<y\}} \overline{\Pi}_j^{+}(x+y)\, F_{j,i}^{-}(\mathrm{d}y) + d_j^{+} f_{j,i}^{-}(-x) \right)}_{=\chi_j^{+} * \widetilde{F}_{j,i}^{-}(\mathrm{d}x)/\,\mathrm{d}x}$$

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$\boldsymbol{\pi}$-compatibility is **necessary** for $\boldsymbol{\pi}$-friendship $\rightsquigarrow$ There are **no** MAP philanthropists!

The Theorem of friends

- denote by $\boldsymbol{\Pi}^{\pm}$ the Lévy measure matrix s.t. $\boldsymbol{\Pi}^{\pm}_{i,i} = \Pi^{\pm}_{i}$ and $\boldsymbol{\Pi}^{\pm}_{i,j} = q^{\pm}_{i,j} F^{\pm}_{i,j}$
- let $\partial\boldsymbol{\Pi}^{\pm}$ be the Lévy density matrix s.t. $\partial\boldsymbol{\Pi}_{i,j}$ is the density of the absolutely continuous part of $\boldsymbol{\Pi}^{+}_{i,j}$

Theorem (Döring, T. and Watson, 2024)

(H^+, J^+) and (H^-, J^-) are $\boldsymbol{\pi}$-friends if, and only if, they are $\boldsymbol{\pi}$-compatible and the matrix-valued function

$$\boldsymbol{\Upsilon}(x) = \begin{cases} \left(\int_{x+}^{\infty} \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \left(\overline{\boldsymbol{\Pi}}^{-}(y - x) - \boldsymbol{\Psi}^{-}(0) \right)^{\top} \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Pi}^{+}(\mathrm{d}y) + \boldsymbol{\Delta}_{d}^{-} \partial\boldsymbol{\Pi}^{+}(x) \right), & x > 0, \\ \left(\int_{(-x)+}^{\infty} \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \left(\boldsymbol{\Pi}^{-}(\mathrm{d}y) \right)^{\top} \boldsymbol{\Delta}_{\boldsymbol{\pi}} \left(\overline{\boldsymbol{\Pi}}^{+}(y + x) - \boldsymbol{\Psi}^{+}(0) \right) + \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \left(\boldsymbol{\Delta}_{d}^{+} \partial\boldsymbol{\Pi}^{-}(-x) \right)^{\top} \boldsymbol{\Delta}_{\boldsymbol{\pi}} \right), & x < 0, \end{cases}$$

is a.e. equal to a function decreasing on $(0, \infty)$ and increasing on $(-\infty, 0)$.

If they are $\boldsymbol{\pi}$-friends, then $\boldsymbol{\Upsilon}$ is a.e. the right/left tail of the Lévy measure matrix of the bonding MAP.

Fellowship

We call (H^+, J^+) and (H^-, J^-) $\boldsymbol{\pi}$-fellows if they have decreasing Lévy density matrices $\partial\boldsymbol{\Pi}^+$ and $\partial\boldsymbol{\Pi}^-$ on $(0, \infty)$, and the matrix functions

$$-\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\overline{\boldsymbol{\Pi}}^+(x) + \boldsymbol{\Delta}_d^-\partial\boldsymbol{\Pi}^+(x), \quad x > 0,$$

and

$$-\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^+(0)^\top\boldsymbol{\Delta}_{\boldsymbol{\pi}}\overline{\boldsymbol{\Pi}}^-(x) + \boldsymbol{\Delta}_d^+\partial\boldsymbol{\Pi}^-(x), \quad x > 0,$$

are decreasing.

Note: Any two Lévy fellows are Lévy philanthropists and therefore friends.

Properties of fellowship

Recall:

H^+ is a Lévy philanthropist $\iff$ H^+ has a decreasing Lévy density $\iff$ H^+ is friends with any pure drift

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Lemma (Döring, T. and Watson, 2024)

A MAP subordinator (H^+, J^+) with decreasing Lévy density matrix is a $\boldsymbol{\pi}$-friend of a pure drift MAP (H^-, J^-) (that is, $H_t^- = \int_0^t d_{J_s^-}^- \, \mathrm{d}s$) if, and only if, they are $\boldsymbol{\pi}$-compatible $\boldsymbol{\pi}$-fellows.

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Recall: Any two Lévy philanthropists are friends.

Properties of fellowship

Recall:
H^+ is a Lévy philanthropist $\iff$ H^+ has a decreasing Lévy density $\iff$ H^+ is friends with any pure drift

Lemma (Döring, T. and Watson, 2024)

A MAP subordinator (H^+, J^+) with decreasing Lévy density matrix is a π-friend of a pure drift MAP (H^-, J^-) (that is, $H_t^- = \int_0^t d_{J_s^-}^- \, \mathrm{d}s$) if, and only if, they are π-compatible π-fellows.

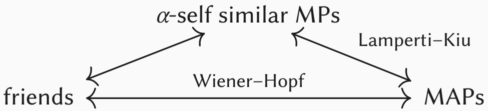
Recall: Any two Lévy philanthropists are friends.

Theorem (Döring, T. and Watson, 2024)

Any two π-compatible π-fellows are π-friends.

Examples

- only known MAP WH-factorisation is from Kyprianou's deep factorisation of the stable process[7]



- to generate explicit friendships it is crucial to find manageable criteria for $\boldsymbol{\pi}$-compatibility
- we give such criteria in two cases
  1. at least one of the putative friends is a pure drift
  2. both candidates have zero drift parts and no transitional atoms (i.e., $q^{\pm}_{i,j}F^{\pm}_{i,j}(\{0\}) = 0$)
- the first case allows us to give a general construction principle for spectrally one-sided MAPs and modulated Brownian motions starting from the WH-factors
- combining the compatibility criteria from both cases, we construct MAPs jumping in both directions (Markov modulated double exponential jump diffusions)

[7] A.E. Kyprinaou (2016). Deep factorisation of the stable process. *Electron. J. Probab.*

Example for one-sided jumps

absolutely monotone Lévy densities

$$\Pi_i^+(\mathrm{d}x) = \int_{\mathbb{R}_+} \mathrm{e}^{-xy}\, \mu_i^+(\mathrm{d}y)\, \mathrm{d}x, \quad x > 0, i = 1, 2,$$

with representing measures μ_i^+ supported on (a_i^+, ∞) for some $a_i^+ > 0$ such that

1. $d_1^+ + \int_{0+}^{\infty} \overline{\Pi}_1^+(x)\,\mathrm{d}x = \frac{\pi(2)}{\pi(1)} \frac{q_{2,1}^+ d_2^-}{q_{1,2}^-}, \quad d_2^+ + \int_{0+}^{\infty} \overline{\Pi}_2^+(x)\,\mathrm{d}x = \frac{\pi(1)}{\pi(2)} \frac{q_{1,2}^+ d_1^-}{q_{2,1}^-};$
2. $a_i^+\left(1 + \frac{d_i^-}{q_{i,j}^-} a_i^+\right) > \frac{q_{j,i}^-}{d_j^-}$
3. $F_{i,j}^+(\mathrm{d}x)|_{(0,\infty)} = f_{i,j}^+(x)\,\mathrm{d}x$ such that

$$f_{1,2}^+(x) = \frac{\pi(2)}{\pi(1)} \frac{q_{2,1}^-}{q_{1,2}^+ d_1^-} \overline{\Pi}_2^+(x), \quad f_{2,1}^+(x) = \frac{\pi(1)}{\pi(2)} \frac{q_{1,2}^-}{q_{2,1}^+ d_2^-} \overline{\Pi}_1^+(x),$$

Then (H^+, J^+) and the pure drift MAP (H^-, J^-) are friends with Lévy measure matrix

$$\boldsymbol{\Pi}(\mathrm{d}x) = \begin{bmatrix} \mathbf{1}_{\{x>0\}} q_{1,2}^- \int_{\mathbb{R}_+} \left(1 + \frac{d_1^- y}{q_{1,2}^-} - \frac{q_{2,1}^-}{d_2^- y}\right) \mathrm{e}^{-xy}\, \mu_1^+(\mathrm{d}y)\,\mathrm{d}x & \frac{\pi(2)}{\pi(1)} q_{2,2}^- \left\{\left(q_{2,2}^+ + q_{1,1}^- \frac{d_2^+}{d_1^-}\right)\delta_0(\mathrm{d}x) + \mathbf{1}_{\{x>0\}} \frac{q_{1,1}^-}{d_1^-} \int_{\mathbb{R}_+} \frac{\mathrm{e}^{-xy}}{y}\, \mu_2^+(\mathrm{d}y)\,\mathrm{d}x\right\} \\ \frac{\pi(1)}{\pi(2)} q_{1,1}^- \left\{\left(q_{1,1}^+ + q_{2,2}^- \frac{d_1^+}{d_2^-}\right)\delta_0(\mathrm{d}x) + \mathbf{1}_{\{x>0\}} \frac{q_{2,2}^-}{d_2^-} \int_{\mathbb{R}_+} \frac{\mathrm{e}^{-xy}}{y}\, \mu_1^+(\mathrm{d}y)\,\mathrm{d}x\right\} & \mathbf{1}_{\{x>0\}} q_{2,1}^- \int_{\mathbb{R}_+} \left(1 + \frac{d_2^- y}{q_{2,1}^-} - \frac{q_{1,2}^-}{d_1^- y}\right) \mathrm{e}^{-xy}\, \mu_2^+(\mathrm{d}y)\,\mathrm{d}x \end{bmatrix}.$$

Uniqueness of the MAP Wiener–Hopf factorisation

Question

- If $(H^{\pm}, J^{\pm})$ are friends, are they equal in law to the ascending/descending ladder height processes of their bonding MAP?
- This is a question of uniqueness of the MAP Wiener–Hopf factorisation: given two factorisations of the same MAP matrix exponent, are the factors equal up to premultiplication by a diagonal matrix with positive entries?

Uniqueness of the MAP Wiener–Hopf factorisation – partial answer

- $\mathcal{A}_0$ is the class of matrix exponents of irreducible and finite mean MAP subordinators
- $\mathcal{A}_\infty$ is the class of MAP subordinator exponents Ψ such that for any i

$$\lim_{\theta \to \pm\infty} |\psi_i(\theta)| = \infty$$

Theorem (Döring, T. and Watson (2024))

Let $(H^\pm, J^\pm)$ be irreducible $\boldsymbol{\pi}$-friends s.t.

- the bonding MAP is irreducible, and
- $\boldsymbol{\Psi}^\pm \in \mathcal{A}_0 \cap \mathcal{A}_\infty$, and the exponents of the ladder height processes of the bonding MAP belong to $\mathcal{A}_0 \cap \mathcal{A}_\infty$.

Then $(H^\pm, J^\pm)$ are versions of the ladder height processes of their bonding MAP.

Some open questions

- Vigon's analysis for Lévy processes demonstrates that if at least one of the factors is unkilled,

$$-\psi^-(-\theta)\psi^+(\theta) = \mathrm{i}a\theta - \tfrac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x \mathbf{1}_{[-1,1]}(x))\, \nu(\mathrm{d}x),$$

  where ν is a signed measure without atom at 0 such that $|\nu|$ integrates $x \mapsto 1 \wedge x^2$
  - $X \sim \mu$ is called quasi-infinitely divisible[8] if there is an infinitely divisible and independent r.v. Y s.t. $X + Y$ is infinitely divisible.
  - then $\hat{\mu} = \mathrm{e}^{\psi}$, where ψ has a Lévy–Khintchine representation with a signed measure. Conversely, e^{ψ} is not necessarily a characteristic function $\rightsquigarrow$ sufficient conditions for $-\psi^-(-\cdot)\psi^+$ to generate a quasi-infinitely divisible distribution?
- MAP-analogue to hypergeometric Lévy processes?
- Friendships from inverted WH-factorisations

$$\underbrace{\boldsymbol{\Psi}(\theta)^{-1}}_{\text{“=”}\,\mathscr{F}\mathbf{U}(\theta)} = -\underbrace{\boldsymbol{\Psi}^+(\theta)^{-1}}_{\text{“=”}\,\mathscr{F}\mathbf{U}^+(\theta)}\, \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \underbrace{(\boldsymbol{\Psi}^-(-\theta)^\top)^{-1}}_{\text{“=”}\,\mathscr{F}\mathbf{U}^-(-\theta)^\top} \boldsymbol{\Delta}_{\boldsymbol{\pi}}$$

[8] A. Lindner, K. Pan, K. Sato (2018). On quasi-infinitely divisible distributions. *Trans. Amer. Math. Soc.*

Some open questions

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$$-\psi^-(-\theta)\psi^+(\theta) = \mathrm{i}a\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x \mathbf{1}_{[-1,1]}(x))\, \nu(\mathrm{d}x),$$

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Thank you for your attention!

[8] A. Lindner, K. Pan, K. Sato (2018). On quasi-infinitely divisible distributions. *Trans. Amer. Math. Soc.*