Change point and change interface estimation for a stochastic heat equation

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based on joint work with Markus Reiß, Claudia Strauch and Anton Tiepner

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Some generalities on statistics fors SPDEs

- Let $-A_\vartheta$ be a non-negative self-adjoint operator on $L^2(\Lambda)$ for some domain $\Lambda \subset \mathbb{R}^d$ and consider the SPDE
$$\begin{cases} \mathrm{d}X(t) = A_\vartheta X(t)\,\mathrm{d}t + \mathrm{d}W(t), & t \in (0, T], \\ X(0) = X_0 \in L^2(\Lambda), & \\ X(t)|_{\partial\Lambda} = 0, & t \in (0, T], \end{cases}$$
where W is a cylindrical Brownian motion (that is, a random linear transformation $z \mapsto (W_z(t))_{t\in[0,T]} =: (\langle W(t), z\rangle)_{t\in[0,T]}$ on L^2 s.t. for any z the rhs is a BM with variance $\|z\|_{L^2}^2$ and $\mathbb{E}[\langle W(t), z\rangle\langle W(s), z'\rangle] = t \wedge s\langle z, z'\rangle$)

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- for $S_\vartheta(t) = \exp(tA_\vartheta)$ the mild solution process is defined by
$$X(t) = S_\vartheta(t)X_0 + \int_0^t S_\vartheta(t-s)\,\mathrm{d}W(s), \quad t \in [0, T],$$
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- the mild solution is a weak solution in the sense

$$\langle X(t), z\rangle = \langle X_0, z\rangle + \int_0^t \langle X(s), A_\vartheta z\rangle\,\mathrm{d}s + \langle W(t), z\rangle, \quad z \in D(A_\vartheta).$$

- spectral observations: provided A_ϑ has an orthonormal eigenbasis (e_j) that is independent of ϑ (think $A_\vartheta = \vartheta\Delta$), observe ($j = 1, \dots, n$, $t \in [0, T]$)

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$$d\langle X(t), K_{\delta,i}\rangle = \langle X(t), A_\vartheta K_{\delta,i}\rangle \, dt + \langle K_{\delta,i}, dW_t\rangle$$

$\rightsquigarrow$ observations are generalized Itô processes (but not independent for $i \neq j$), asymptotics: $\delta \to 0$, n may be fixed or increase with δ^{-1}

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A change point model for a stochastic heat equation

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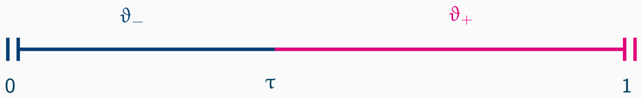
for a weighted Laplacian $\Delta_\vartheta \coloneqq \nabla\vartheta\nabla$, with discontinuous diffusivity

$$\vartheta(x) = \vartheta_- \mathbf{1}_{(0,\tau)}(x) + \vartheta_+ \mathbf{1}_{[\tau,1)}(x), \quad x,\tau \in (0,1), 0 < \vartheta_- \wedge \vartheta_+.$$



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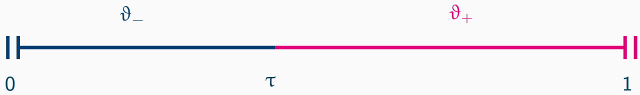


Classical heat kernel bounds for analytic semigroup $(S_\vartheta(t) = \exp(t\Delta_\vartheta))_{t\in[0,T]}$ implies that

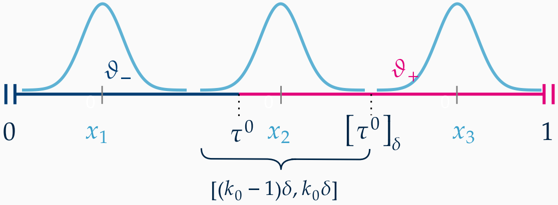
$$X(t) = \int_0^t S_\vartheta(t-s)\,\mathrm{d}W_s, \quad t \in [0,T]$$

lives in $L^2((0,1))$ and we have

$$\langle X(t), z\rangle = \int_0^t \langle X(s), \Delta_\vartheta z\rangle\,\mathrm{d}s + \langle W(t), z\rangle, \quad z \in D(\Delta_\vartheta) = \left\{u \in H_0^1((0,1)) : \vartheta\nabla u \in H^1((0,1))\right\}.$$

Estimation approach

- let $K\colon \mathbb{R} \to \mathbb{R}$ be a smooth kernel with $\operatorname{supp} K \subset [-1/2, 1/2]$, $\|K\|_{L^2} = 1$ and for $\delta = n^{-1}$, $x_i = (i - 1/2)\delta$ $(i \in \{1, \dots, \delta^{-1}\})$, define $K_{\delta,i} = \delta^{-1/2} K(\delta^{-1}(x - x_i))$
- local observations $(X_{\delta,i}(t))_{t\in[0,T]} = (\langle X(t), K_{\delta,i}\rangle)_{t\in[0,T]}$ and $(X^{\Delta}_{\delta,i}(t))_{t\in[0,T]} = (\langle X(t), \Delta K_{\delta,i}\rangle)_{t\in[0,T]}$



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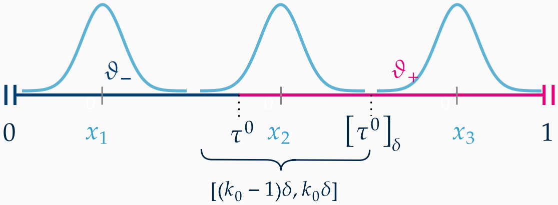
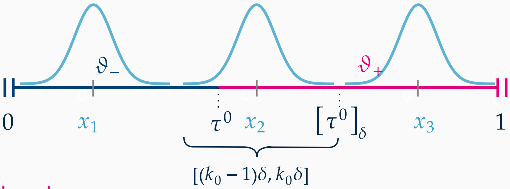


- if $i \neq k_0$, then $X_{\delta,i}$ solves $\mathrm{d}X_{\delta,i}(t) = \vartheta(x_i) X^{\Delta}_{\delta,i}(t)\,\mathrm{d}t + \mathrm{d}B_{\delta,i}(t)$ for independent BMs $(B_{\delta,i}, i \in [\delta^{-1}])$
- we define a modified local log-likelihood by

$$\ell_{\delta,i}(\vartheta_-, \vartheta_+, \vartheta_\circ, k) := \vartheta_{\delta,i}(k) \int_0^T X^{\Delta}_{\delta,i}(t)\,\mathrm{d}X_{\delta,i}(t) - \frac{\vartheta_{\delta,i}(k)^2}{2} \int_0^T X^{\Delta}_{\delta,i}(t)^2\,\mathrm{d}t, \quad \vartheta_{\delta,i}(k) := \begin{cases} \vartheta_-, & i < k, \\ \vartheta_\circ, & i = k \\ \vartheta_+, & i > k \end{cases}$$



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- CUSUM-approach: $(\widehat{\vartheta}_-, \widehat{\vartheta}_+, \widehat{\vartheta}_\circ, \widehat{\tau}) := (\widehat{\vartheta}_-, \widehat{\vartheta}_+, \widehat{\vartheta}_\circ, \widehat{k}\delta)$, where

$$(\widehat{\vartheta}_-, \widehat{\vartheta}_+, \widehat{\vartheta}_\circ, \widehat{k}) := \operatorname*{arg\,max}_{(\vartheta_-, \vartheta_+, \vartheta_\circ, k)} \sum_{i \in [\delta^{-1}]} \ell_{\delta,i}(\vartheta_-, \vartheta_+, \vartheta_\circ, k)$$

$$= \operatorname*{arg\,min}_{(\vartheta_-, \vartheta_+, \vartheta_\circ, k)} \left\{ \frac{1}{2} \sum_{i=1}^{\delta^{-1}} (\vartheta_{\delta,i}(k) - \vartheta^0_{\delta,i})^2 I_{\delta,i} - \sum_{i=1}^{\delta^{-1}} (\vartheta_{\delta,i}(k) - \vartheta^0_{\delta,i}) M_{\delta,i} - \vartheta_{\delta,k_0}(k) R_{\delta,k_0}(\vartheta^0_\circ) \right\},$$

for

$$M_{\delta,i} := \int_0^T X^\Delta_{\delta,i}(t)\, \mathrm{d}B_{\delta,i}(t), \quad I_{\delta,i} := \int_0^T X^\Delta_{\delta,i}(t)^2\, \mathrm{d}t,$$

and $R_{\delta,k_0}(\vartheta^0_\circ)$ is an error term resulting from $K_{\delta,k_0} \notin D(\Delta_\vartheta)$ in general

Basic estimates

Lemma (Reiß, Strauch and T., 2023+)

- For any $i \in [\delta^{-1}] \setminus \{k_0\}$,

$$\mathbb{E}[I_{\delta,i}] = \frac{T}{2\vartheta(x_i)} \|K'\|_{L^2}^2 \delta^{-2} + \mathcal{O}(1),$$

  and, moreover, $\mathbb{E}[I_{\delta,k_0}] \sim \delta^{-2}$;
- for any vector $\alpha \in \mathbb{R}^n$ s.t. $\alpha_{k_0} = 0$,

$$\mathrm{Var}\Big(\sum_{i=1}^{\delta^{-1}} \alpha_i I_{\delta,i}\Big) \leqslant \frac{T}{2\underline{\vartheta}^3} \delta^{-2} \|\alpha\|_{\ell^2}^2 \|K'\|_{L^2}^2;$$

- For $\eta \coloneqq \vartheta_+^0 - \vartheta_-^0$,

$$\mathbb{E}[|R_{\delta,k_0}(\vartheta_\circ)|] \lesssim \delta^{-2}, \quad \mathrm{Var}(R_{\delta,k_0}(\vartheta_\circ)) \lesssim \delta^{-2},$$

  and $\exists \vartheta_\circ^0$ s.t. $|\mathbb{E}[R_{\delta,k_0}(\vartheta_\circ^0)]| \leqslant \delta^{-1}$.

Main observation: $\sum_{i=1}^{\delta^{-1}} \alpha_i(I_{\delta,i} - \mathbb{E}[I_{\delta,i}])$ belongs to second Wiener chaos for an appropriate isonormal Gaussian process associated to $(X_i^{\Delta}(t))_{t\in[0,T],i\in[\delta^{-1}]}$ $\rightsquigarrow$ verify conditions for Bernstein-type concentration inequality from Nourdin and Viens (2009)[2]

Proposition (Reiß, Strauch and T., 2023+)

Let $\alpha \in \mathbb{R}_+^n \setminus \{0\}$ s.t. $\alpha_{k_0} = 0$. Then, for any $z > 0$, we have

$$\mathbb{P}\Big(\Big|\sum_{i=1}^{n} \alpha_i(I_{\delta,i} - \mathbb{E}[I_{\delta,i}])\Big| \geqslant z\Big) \leqslant 2\exp\Big(-\frac{\underline{\vartheta}^2}{4\|\alpha\|_\infty}\frac{z^2}{z+\sum_{i=1}^n \alpha_i \mathbb{E}[I_{\delta,i}]}\Big).$$

[2] I. Nourdin and F.G. Viens (2009). Density formula and concentration inequalities with Malliavin calculus. *Electron. J. Prob.*, 14:no. 78, 2287–2309.

Define the jump height $\eta \coloneqq \vartheta_+^0 - \vartheta_-^0$.

Theorem (Reiß, Strauch and T., 2023+)

Suppose that $\chi^0(\delta) \underset{\delta\to 0}{\longrightarrow} \chi^*$ and that $|\eta| \geqslant \underline{\eta}$ for all $\delta \in 1/\mathbb{N}$. Then,

$$\widehat{\tau} - \tau^0 = \mathcal{O}_{\mathbb{P}}(\delta) \quad \text{and} \quad \widehat{\vartheta}_\pm - \vartheta_\pm^0 = \mathcal{O}_{\mathbb{P}}(\delta^{3/2}).$$

- the estimation rate for τ^0 is the same as in classical discrete change point models
- the estimation rate for $\vartheta_\pm^0$ is the same as the minimax optimal rate[3] for parametric estimation from multiple local measurements in the model $A_\vartheta = \vartheta\Delta$

[3] Altmeyer, Tiepner, Wahl (2023+) Optimal parameter estimation for linear SPDEs from multiple measurements

Limit theorem for vanishing jump height

- for the previous consistency result it was crucial that the jump height η does not vanish
- assume now that $\eta \xrightarrow[\delta\to 0]{} 0$ and that the nuisance parameters $\vartheta_\pm^0 = \vartheta_\pm^0(\delta)$ are known
- CUSUM estimator: $\widehat{\tau} = \widehat{k}\delta$, where

$$\widehat{k} := \underset{k=1,\dots,\delta^{-1}}{\arg\max} \sum_{i=1}^{k} \left(\vartheta_-^0 \int_0^T X_{\delta,i}^{\Delta}(t)\,\mathrm{d}X_{\delta,i}(t) - \frac{(\vartheta_-^0)^2}{2}\int_0^T X_{\delta,i}^{\Delta}(t)^2\,\mathrm{d}t\right) + \sum_{i=k+1}^{\delta^{-1}} \left(\vartheta_+^0 \int_0^T X_{\delta,i}^{\Delta}(t)\,\mathrm{d}X_{\delta,i}(t) - \frac{(\vartheta_+^0)^2}{2}\int_0^T X_{\delta,i}^{\Delta}(t)^2\,\mathrm{d}t\right)$$

Theorem (Reiß, Strauch and T., 2023+)

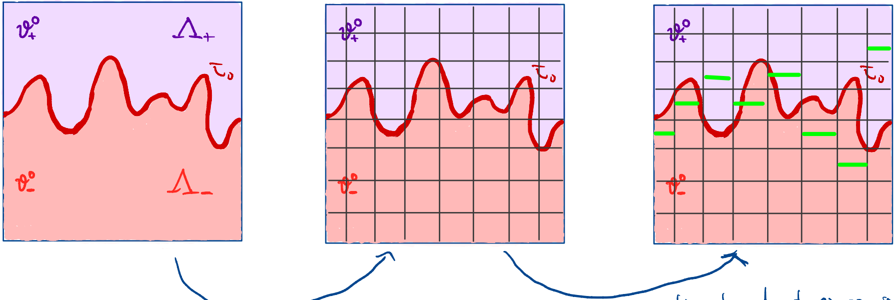
Assume $\eta = o(\delta)$ and $\delta^{3/2} = o(\eta)$. Then, for a two-sided Brownian motion $(B^{\leftrightarrow}(h),\ h\in\mathbb{R})$, we have

$$\frac{\eta^2}{\delta^3}\frac{T\|K'\|_{L^2}^2}{2\vartheta^*}(\widehat{\tau}-\tau) \xrightarrow{\mathrm{d}} \underset{h\in\mathbb{R}}{\arg\min}\left\{B^{\leftrightarrow}(h) + \frac{|h|}{2}\right\}, \qquad \text{as } \delta\to 0.$$

$dX_t = \Delta_{\vartheta^0} X_t \, dt + dW_t$, $\quad \vartheta^0(x) = \vartheta^0_- \mathbb{1}_{\Lambda_-}(x) + \vartheta^0_+ \mathbb{1}_{\Lambda_+}(x), \; x \in \Lambda \supset [0,1]^d$

$$\Delta_{\vartheta^0} u = \nabla \cdot \vartheta^0 \nabla u = \sum_{i=1}^{d} \partial_i (\vartheta^0 \partial_i u)$$

$$\partial\Lambda_- \cap \partial\Lambda_+ \cap [0,1]^d = \{ (x, \tau^0(x)) : x \in [0,1]^{d-1} \}$$



δ^{-d} measurement locations,

loc. meas: $\langle X(t), K_{\delta,i} \rangle$, $\langle X(t), \Delta K_{\delta,i} \rangle$
over time interval $[0,T]$

estimate best piecewise constant approximation $\hat{\tau}$ based on CUSUM approach

Convergence result in the multivariate case

Theorem (Tiepner and T., 2024+)

Let $(\widehat{\vartheta}_-, \widehat{\vartheta}_+, \widehat{\tau})$ be the CUSUM estimator of $(\vartheta^0_-, \vartheta^0_+, \tau^0)$, designed under the assumption that we know two sets $\Theta_\pm \subset [\underline{\vartheta}, \overline{\vartheta}]$ that contain $\vartheta^0_\pm$ and are $\underline{\eta}$-separated, $\underline{\eta} > 0$. If τ^0 is β-Hölder continuous on $[0,1]^{d-1}$, then

$$\mathbb{E}\Big[\|\widehat{\tau} - \tau^0\|_{L^1([0,1]^{d-1})}\Big] \lesssim \delta^\beta.$$

Given the necessary identifiability assumption $\|\tau^0\|_{L^1([0,1]^{d-1})} \in (0,1)$, $(\widehat{\vartheta}_-, \widehat{\vartheta}_+)$ is a consistent estimator of $(\vartheta^0_-, \vartheta^0_+)$.

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In terms of the number of measurement location $N = \delta^{-d}$, the change profile estimation rate is given by

$$N^{-\beta/d} \lesseqgtr \underbrace{N^{\frac{-\beta}{2\beta+d-1}}}_{\text{nonparametric rate}}, \quad \beta \gtreqless 1/2.$$

This is the same rate that is observed in image reconstruction problems with regular design.

Summary

- for a stochastic heat equation with piecewise constant diffusivity, we construct a simultaneous M-estimator for the conductivities $\vartheta_{\pm}^{0}$ and the change point τ^{0} from multiple local measurements
- in case of non-vanishing jump height, we show that

$$\widehat{\tau} - \tau^{0} = \mathcal{O}_{\mathbb{P}}(\delta) \quad \text{and} \quad \widehat{\vartheta}_{\pm} - \vartheta_{\pm}^{0} = \mathcal{O}_{\mathbb{P}}(\delta^{3/2})$$

- in case of vanishing jump height and known parameters $\vartheta_{\pm}^{0}$ we construct a change point estimator $\widehat{\tau}$ obeying the limit theorem

$$\frac{\eta^2}{\delta^3}\frac{T\|K'\|_{L^2}^2}{2\vartheta^*}(\widehat{\tau} - \tau^{0}) \xrightarrow{\mathrm{d}} \operatorname*{arg\,min}_{h\in\mathbb{R}}\left\{B^{\leftrightarrow}(h) + \frac{|h|}{2}\right\}, \qquad \text{as } \delta \to 0,$$

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Thank you for your attention!